Breidd margfeldnifalls

Höfum áhuga á $N \sim 10^{20}$ eindum

Finnum nálgun fyrir

$$g(N,s) = \frac{N!}{(\frac{1}{2}N+s)!\,(\frac{1}{2}N-s)!} = \frac{N!}{N_\uparrow!\,N_\downarrow!}$$

útleiðsla er í viðbót A í bók

Notum nálgun Stirlings fyrir $N \gg 1$

$$N! \simeq \sqrt{2\pi N}\, N^N \exp\left\{-N + \frac{1}{12N} + \frac{1}{288N^2} + \cdots\right\}$$

og fyrir $x \ll 1$

$$\ln(1+x) \simeq x - \frac{x^2}{2} + \cdots$$

$$\rightarrow \ln N! \simeq \frac{1}{2}\ln(2\pi) + \left(N+\frac{1}{2}\right)\ln N - N$$

$$\ln g = \ln N! - \ln N_\uparrow! - \ln N_\downarrow! \qquad (*)$$

$$N = N_\uparrow + N_\downarrow$$

$$\ln N! \simeq \frac{1}{2}\ln(2\pi) + (N + \tfrac{1}{2})\ln N - N$$

$$\simeq \frac{1}{2}\ln(2\pi/N) + (N_\uparrow + N_\downarrow + \tfrac{1}{2})\ln N - N_\uparrow - N_\downarrow \quad + \frac{1}{2}\ln N$$

$$\simeq \frac{1}{2}\ln\left(\frac{2\pi}{N}\right) + (N_\uparrow + \tfrac{1}{2} + N_\downarrow + \tfrac{1}{2})\ln N - (N_\uparrow + N_\downarrow)$$

Notum i (*)

$$\ln g \simeq \frac{1}{2}\ln\left(\frac{1}{2\pi N}\right) - (N_\uparrow + \tfrac{1}{2})\ln\left(\frac{N_\uparrow}{N}\right) - (N_\downarrow + \tfrac{1}{2})\ln\left(\frac{N_\downarrow}{N}\right)$$

$$\ln\left(\frac{N_\uparrow}{N}\right) = \ln\left[\frac{\frac{1}{2}N + s}{N}\right] = \ln\left\{\frac{1}{2}\cdot\left(1 + \frac{2s}{N}\right)\right\}$$

$$= -\ln 2 + \ln\left(1 + \frac{2s}{N}\right)$$

$$\simeq -\ln 2 + \left(\frac{2s}{N}\right) - \left(\frac{2s^2}{N^2}\right)$$

á sama hátt fæst

$$\ln\left(\frac{N_\downarrow}{N}\right) = \ln\left\{\frac{\frac{1}{2}N - s}{N}\right\} \simeq -\ln 2 - \left(\frac{2s}{N}\right) - \left(\frac{2s^2}{N^2}\right)$$

$$\rightarrow \ln g \simeq \frac{1}{2}\ln\left(\frac{2}{\pi N}\right) + N\ln 2 - \frac{2s^2}{N}$$

eða

$$g(N,s) \simeq g(N,0) \exp\left[-\frac{2s^2}{N}\right]$$

og $$g(N,0) \simeq \sqrt{\frac{2}{\pi N}}\, 2^N$$

sem sé **Gauss-dreifing**

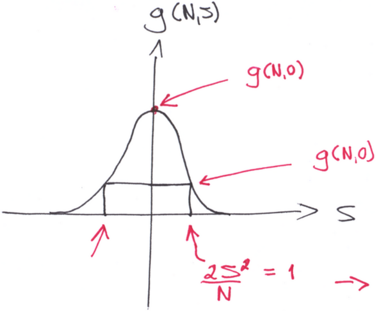
Heildarfjöldinn er réttur því

$$\int_{-\infty}^{\infty} ds\, g(N,s) = 2^N$$

Munum líka að

$$g(N,0) = \frac{N!}{(\frac{1}{2}N)!\,(\frac{1}{2}N)!}$$

nákvæmlega



$$\frac{2s^2}{N} = 1 \rightarrow s^2 = \frac{N}{2}$$

$$s = \sqrt{\frac{N}{2}}$$

eða $\frac{s}{N} = \sqrt{\frac{1}{2N}}$: Hlutfallsleg breidd $g(N,s)$

fyrir $N \approx 10^{22}$ fæst

$$\frac{s}{N} \sim 10^{-11}$$

Mjög grönn dreifing fyrir

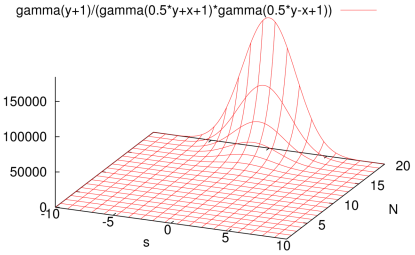
$$\frac{s}{N} \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$$

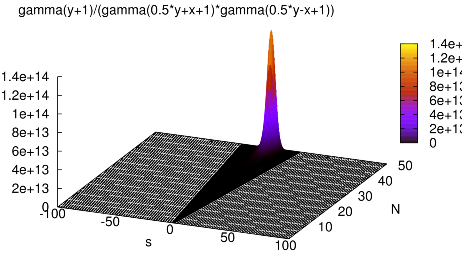
Athugasemdir

Hér er ekkert tekið til um orku ástandanna

mesti fjöldi þeirra er fyrir $s=0$ og þar nærri

Ef öll ástöndin væru jafn líkleg væri langlíklegast að finna kerfið í ástandi með $s \simeq 0$


gamma(y+1)/(gamma(0.5*y+x+1)*gamma(0.5*y-x+1))
150000
100000
50000
0
-10
-5
0
5
10
s
5
10
15
20
N


gamma(y+1)/(gamma(0.5*y+x+1)*gamma(0.5*y-x+1))
1.4e+14
1.2e+14
1e+14
8e+13
6e+13
4e+13
2e+13
0
-100
-50
0
50
100
s
10
20
30
40
50
N
0

Meðaltöl

Líkindadreifing $P(s)$

$$\rightarrow \sum_s P(s) = 1$$ normun

Meðaltal falls $f(s)$ er

$$\langle f \rangle = \sum_s f(s) P(s)$$

Fyrir margfeldnisfallið $g(N,s)$ gildir

$$\sum_s g(N,s) = 2^N$$ heildar fjöldi ástanda

Ef öll ástöndin eru jafn líkleg verð líkindadreifingin

$$P(s) = \frac{g(N,s)}{2^N}$$

og meðaltalið

$$\langle f \rangle = \sum_s f(s) P(N,s)$$

því er t.d.

$$\langle s^2 \rangle = \sum_s s^2 P(N,s)$$

$$= \sqrt{\frac{2}{\pi N}}\, 2^N \int_{-\infty}^{\infty} ds\, s^2 e^{-\frac{2s^2}{N}} / 2^N$$

$$\langle s^2 \rangle = \sqrt{\frac{2}{\pi N}} \left(\frac{N}{2}\right)^{3/2} \int_{-\infty}^{\infty} dx\, x^2 e^{-x^2}$$

$$= \sqrt{\frac{2}{\pi N}} \left(\frac{N}{2}\right)^{3/2} \sqrt{\frac{\pi}{4}} = \frac{N}{4}$$

eða $\langle (2s)^2 \rangle = \langle (N_\uparrow - N_\downarrow)^2 \rangle = N$

og $\sqrt{\langle (2s)^2 \rangle} = \sqrt{N}$ RMS gildi

$$\mathcal{F} \equiv \frac{\sqrt{\langle (2s)^2 \rangle}}{N} = \frac{1}{\sqrt{N}}$$ hlutfallslegt flökt

verður hverfandi þegar N er mjög stórt

Orka

fyrir spunakerfi í ytra segulsviði gildir

$$U = -\overline{m}\cdot\overline{B}$$

$\overline{m}$ er segulvægi kerfisins

fyrir okkar óvíxlverkandi kerfi gildir

$$U = \sum_{i=1}^{N} U_i = -\overline{B}\cdot\sum_{i=1}^{N}\overline{m}_i$$

$$= -2smB = -MB$$

Ef $M = 2sm$

fjöleinda orkuróf með fast bili milli stiga

$$\Delta E = U(s) - U(s+1)$$
$$= 2mB$$

Dæmi: Hreintóna sveifill

Einn sveifill hefur orkuróf

$$\mathcal{E} = \hbar\omega \cdot s$$

s: skammtatala

$s \in \{0, 1, 2, \ldots \infty\}$

Óendanlega mörg ástönd, en hvert er einfalt

$$g(1, s) = 1$$

N sveiflar, sama tíðni

Viljum finna á hve marga vegu hægt er að deila orkunni

$$\mathcal{E} = \sum_{i=1}^{N} s_i \cdot \hbar\omega = n\hbar\omega$$

má skipta milli sveiflanna

þarf að finna $g(N, n)$: fjölda möguleika til að deila n skömmtum milli N-sveifla fyrst $g(1, \ ) = 1$

$$\rightarrow \sum_{n=0}^{\infty} g(1,n)t^n = \sum_{n=0}^{\infty} t^n = (t^0 + t^1 + t^2 + t^3 \dots)$$

stærðfræðilega

Vitum líka

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$$

N sveiflar

$$(t^0+t^1+t^2+\dots)(t^0+t^1+t^2+\dots)\dots(\ )$$

$$\left(\sum_{s=0}^{\infty} t^s\right)^N = \left(\frac{1}{1-t}\right)^N = \sum_{n=0}^{\infty} g(N,n)t^n$$

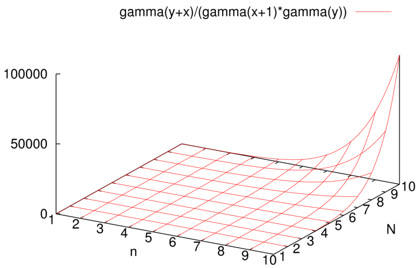
fjöldi liða t^n í N-feldi er fjöldi leiða til að mynda n sem summu N jákvæðra (+0) heiltalna ($N \geq 0$)

$$\rightarrow g(N,n) = \lim_{t\to 0} \frac{1}{n!}\left(\frac{d}{dt}\right)^n \sum_{s=0}^{\infty} g(N,s)t^s$$

$$= \lim_{t\to 0} \frac{1}{n!}\left(\frac{d}{dt}\right)^n (1-t)^{-N}$$

$$= \frac{1}{n!} N(N+1)(N+2)\dots(N+n-1)$$

$$= \frac{(N+n-1)!}{n!\,(N-1)!}$$


gamma(y+x)/(gamma(x+1)*gamma(y))
100000
50000
0
1
2
3
4
5
6
7
8
9
10
n
N