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Jaynes-Cummings model

Consider a system that consists of a completely isolated atom with only two energy levels. These two energy levels correspond to a ground state $|g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and an excited state $|e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Let's denote the energy difference between these two states as $\hbar\omega$ where ω is some angular frequency. We can shift the energy spectrum such that $E_g = -\frac{1}{2}\hbar\omega$ and $E_e = \frac{1}{2}\hbar\omega$. We see that the Hamiltonian of this system can be written as

$$\hat{H}_{\text{atom}} = \frac{\hbar\omega}{2}\hat{\sigma}_z = \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{1}$$

It is easy to check that $\hat{H}_{\text{atom}}|g\rangle = -\frac{1}{2}\hbar\omega$ and $\hat{H}_{\text{atom}}|e\rangle = +\frac{1}{2}\hbar\omega$.

Now consider quantized electromagnetic field in free space. The reader is likely used to treating electromagnetic field field semi-classically where the electromagnetic field is given by the gradient/curl of some simple potential (simple in the sense that it contains no operators). This is often a good approximation but if the electric field is monochromatic (for example a laser) and the energy of a single photon is comparable with some characteristic energy in the system, we have to treat the electric field in a quantized way. We do this by writing the electromagnetic field as an operator. The Hamiltonian of such a field in free space is

$$\hat{H}_{\text{EM}} = \hbar\nu\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \quad , \tag{2}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the photon creation and annihilation operators and ν is the angular frequency of the EM field. Derivation of the form of this Hamiltonian will not be performed here but the interested reader is advised to read the

"Quantization of the electromagnetic field" article on Wikipedia. We can drop the $\frac{1}{2}$ in (2) since a constant addition the energy does not change any dynamics. We can therefore write

$$\hat{H}_{\mathrm{EM}} = \hbar\nu\hat{a}^\dagger\hat{a} \ . \tag{3}$$

Eigenstates of (3) are eigenstates of the photon number operator $\hat{N} = \hat{a}^\dagger\hat{a}$. These states can be written as $|M\rangle = |0\rangle, |1\rangle, |2\rangle, \ldots$ where $\hat{H}_{\mathrm{EM}}|M\rangle = M\hbar\nu|M\rangle$.

We have now written the Hamiltonian for an isolated two level atom (TLA) and free field quantized electric field. If we mix the two systems together the electric part of the EM field will interact with the atom via its dipole moment. We can therefore write the Hamiltonian of the composite system as

$$\hat{H} = \hat{H}_{\mathrm{atom}} + \hat{H}_{\mathrm{EM}} + \hat{H}_{\mathrm{int}} \ , \tag{4}$$

where $\hat{H}_{\mathrm{int}}$ contains the field-atom interaction and is given by

$$\hat{H}_{\mathrm{int}} = \frac{\hbar\Omega}{2}\hat{E}\hat{S} \tag{5}$$

where $\hat{E} = \hat{a} + \hat{a}^\dagger$ is the electric field operator, $\hat{S} = \hat{\sigma}_+ + \hat{\sigma}_-$ is the polarization operator of the atom and $\hbar\Omega$ denotes the strength of the atom-EM interaction, which is dependent on the atom's dipole moment. The operators $\hat{\sigma}_+$ and $\hat{\sigma}_-$ are the Pauli raising and lowering operators given by

$$\hat{\sigma}_+ = |e\rangle\langle g| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \hat{\sigma}_- = |g\rangle\langle e| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ . \tag{6}$$

We can therefore write

$$\hat{H}_{\mathrm{int}} = \frac{\hbar\Omega}{2}(\hat{a} + \hat{a}^\dagger)(\hat{\sigma}_+ + \hat{\sigma}_-) = \frac{\hbar\Omega}{2}(\hat{a}\hat{\sigma}_+ + \hat{a}\sigma_- + \hat{a}^\dagger\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-) \ . \tag{7}$$

The total Hamiltonian is then

$$\hat{H} = \frac{\hbar\omega}{2}\sigma_z + \hbar\nu\hat{a}^\dagger\hat{a} + \frac{\hbar\Omega}{2}(\hat{a}\hat{\sigma}_+ + \hat{a}\sigma_- + \hat{a}^\dagger\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-) \ . \tag{8}$$

The above Hamiltonian is called the Jaynes-Cummings Hamiltonian and it is very important in the quantum theory of radiation and quantum information. The JC model is very popular due to its simplicity. However, in reality there is no such thing as a two level atom and there is always some spread in the photon energy (it is never completely monochromatic). However, we can choose two energy levels of a complicated atom and ignore all the others ones and apply the JC model if the following conditions are met:

1. The two chosen energy levels of the atom are relatively isolated. That is there are no allowed energy levels near the two chosen ones.

2. The photon energy matches the energy spacing between the two chosen levels. That is $\omega \simeq \nu$.

3. There are no selection rules that prohibit the transition between the chosen levels.

4. The atom-field coupling is very small compared with the characteristic energies of the system or $\Omega \ll \omega, \nu$.

Project

Your project is to use a diagonalization method to calculate the energy spectrum of (8). You will use the basis $\{|i, M\rangle\}$ where $|i, M\rangle = |i\rangle \otimes |M\rangle$ and $i \in \{g, e\}$. You need to find a way to number the states in the basis with a single number in order construct matrices. A good idea is to use

$$|g, 0\rangle = |1\rangle, |e, 0\rangle = |2\rangle, |g, 1\rangle = |3\rangle, |e, 1\rangle = |4\rangle, ..., |i, k\rangle = |\mu\rangle \tag{9}$$

In other words you need to construct a bijection (a one to one function)

$$(i, M) \to \mu \ , \mu \in \mathbb{Z}_{+}. \tag{10}$$

and its inverse

$$\mu \to (i, M) \tag{11}$$

The matrix elements of $\hat{H}$ are then calculated using

$$H_{\mu\nu} = \langle\mu|H|\nu\rangle = \langle i, M|H|j, N\rangle \tag{12}$$

The matrix elements of $\hat{H}_{\text{atom}}$ and $\hat{H}_{\text{field}}$ are trivial to compute but the matrix elements of $\hat{H}_{\text{int}}$ are a bit more tricky.

Choose the values of ω, ν and Ω such that the detuning $\delta = |\omega - \nu| / \omega \simeq 1\%$ and $\Omega < \omega/2$. Investigate how the ratio Ω/ω and the value of δ affect the energy spectrum. As for scaling, always use energy in units of $\hbar\omega$.