

3. Project

①

We revisit the damped harmonic oscillator in project 2:

$$i\hbar \dot{g} = [H, g] - \frac{i\hbar}{2} \{[a g, a^\dagger] + [a, g a^\dagger]\} \quad (1)$$

$$H(t) = H_0 + H'(t), \quad H_0 = \hbar\omega \left\{ a^\dagger a + \frac{1}{2} \right\}$$

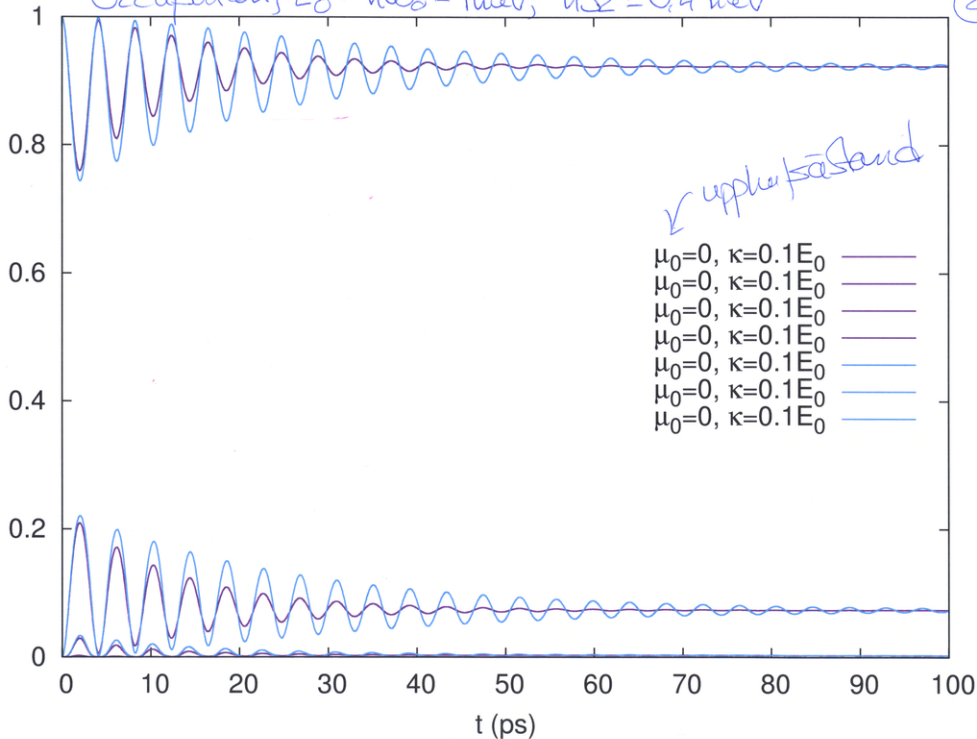
$$H'(t) = \hbar\Omega \{ a^\dagger + a \} \theta(t)$$

There we solved Eq. (1) to find the time-evolution of $g(t)$. With the finite damping we saw that the solution tends to a steady state as $t \rightarrow \infty$.

Occupation, $E_0 = \hbar\omega_0 = 1 \text{ meV}$, $\hbar\Omega = 0.4 \text{ meV}$

(2)

n_μ



The question now is if we can find the steady-state (3)
directly without doing the time-integration?

We can rewrite the dissipative eq. of motion (1) as

$$\dot{g}(t) = -\mathcal{L}[g(t)]$$

In the steady state $\dot{g} = 0$, and we need to solve

$$0 = -\mathcal{L}[g(t)]$$

As we have

$$i\mathcal{L}[g] = \frac{1}{\hbar}[H, g] - \frac{i\kappa}{2\hbar} \{[a, g a^\dagger] + [a, g a^\dagger]\}$$

Our steady state equation can be written as (4)

$$A\mathbf{p} + \mathbf{p}B = C \quad (2)$$

with $C=0$

homogeneous \uparrow

which is Lyapunov's equation, or the Sylvester equation.

Numerically, it is the ~~best~~ practice here to solve it directly, which can be accomplished by applying several LAPACK-routines related to linear algebra transformations and generalized eigenvalue problems.

But for fun let's try a different method here, that will introduce to us some new matrix manipulations worth to know.

In the Matrix Cookbook a solution is given for (2)

in a N^2 -space if the original matrices have dimension N . (5)

$$\text{vec}(p) = (\mathbf{I} \otimes \mathbf{A} + \mathbf{B}^T \otimes \mathbf{I})^{-1} \text{vec}(c)$$

where vectorization of matrices together with the tensorproduct of Kronecker is used.

More cleverly we have now to solve the linear equation

$$\{\mathbf{I} \otimes \mathbf{A} + \mathbf{B}^T \otimes \mathbf{I}\} \text{vec}(p) = \text{vec}(c) = 0$$

↑
here

The best way is thus to use the routine GEEV for non symmetric complex eigenvalue problem to find the complex eigenvalues. There should be a zero-eigenvalue with the steady state p_s as an eigenvector

In the next page we see how the calculation for the steady state agree. (6)

Do this calculation, and find some interesting properties to explore further.

If in Liouville-space we use the notation $\dot{\rho} = -i\mathbb{L}\rho$ then we get now left eigenvectors U and right eigenvectors V

$$\mathbb{L}V = V\lambda$$

$$U\mathbb{L} = \lambda U$$

where λ is a diagonal matrix containing the eigenvalues

U. Hohenester, PRB 81, 155303 (2010)

$$\text{And } VU^\dagger = U^\dagger V = \mathbb{I}$$

(7)

So, in case H is not t -dependent, except through $\Theta(t)$ we have the direct evolution (analytical)

$$\rho(t) = (V \exp[-i\lambda t] U^\dagger) \rho_0$$

in Liouville space

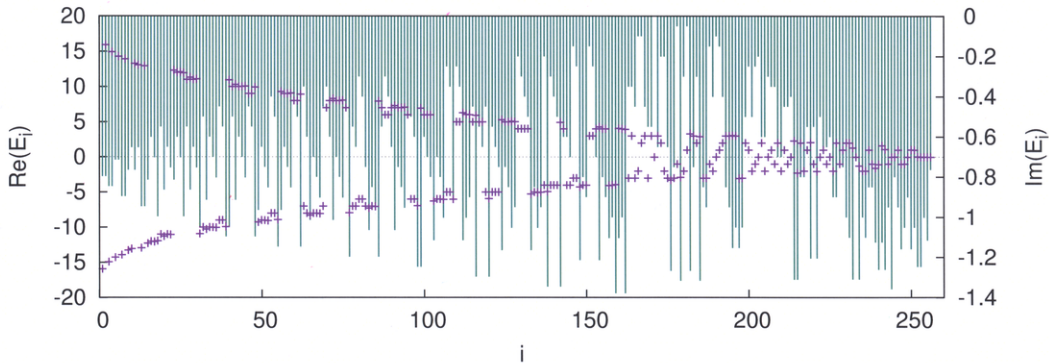
There is a small technical problem when using GEEV due to normalization of eigenvectors, the solution there to will be discussed in lecture

Try this time evolution and compare it to the one in project (2)

The eigen spectrum of the damped oscillator

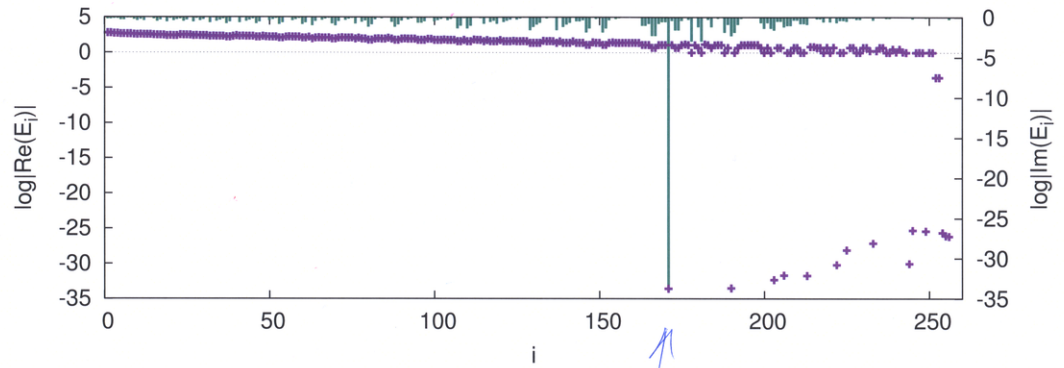
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$$\hbar\omega = 1,0 \text{ meV}, \quad \hbar\Omega = 0,4 \text{ meV}, \quad K = 0,2 \text{ meV}$$



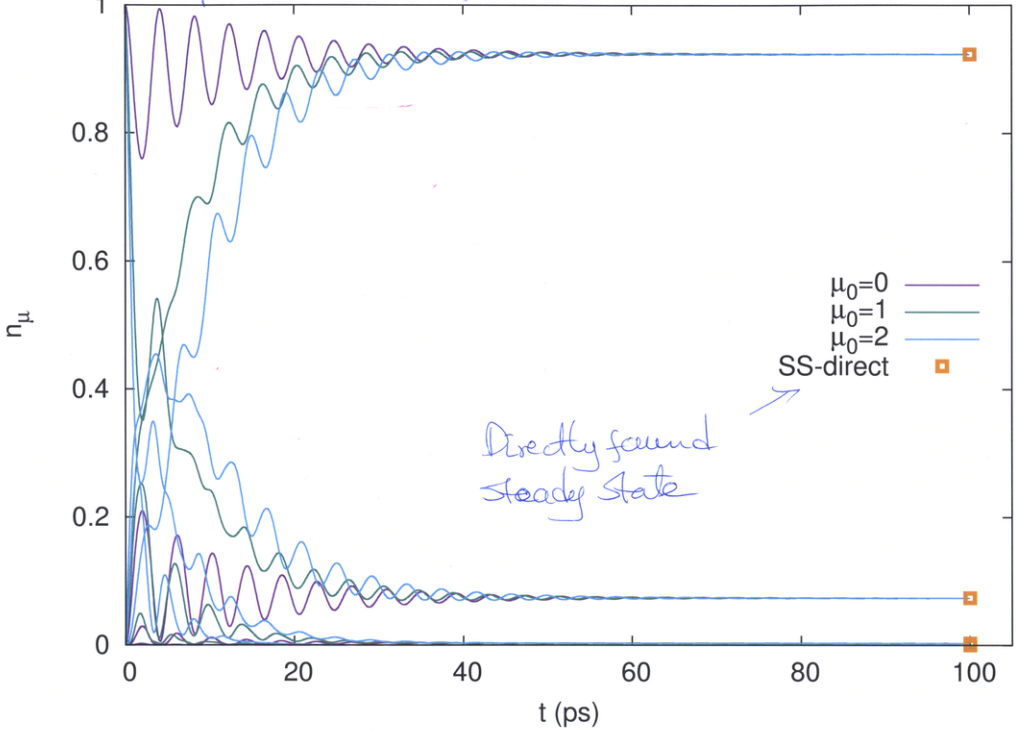
The eigenspectrum of the damped oscillator

$\tau\omega = 1.0 \text{ uel}$, $\tau\Omega = 0.4 \text{ uel}$, $\kappa = 0.2 \text{ uel}$



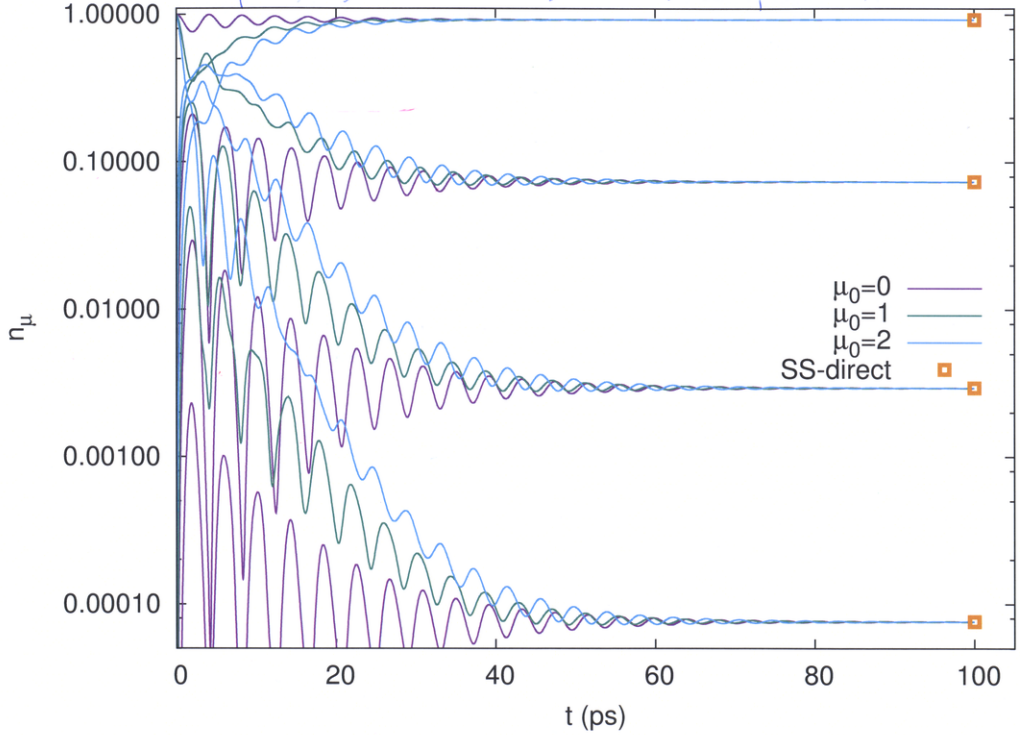
One zero found

Occupation, $t_{res} = 1.0 \text{ meV}$, $t_{\Omega} = 0.4 \text{ meV}$, $\kappa = 0.2 \text{ meV}$



Occupation, $t\omega = 1.0 \text{ meV}$, $t\Delta = 0.4 \text{ meV}$, $K = 0.2 \text{ meV}$

(11)



Occupation, $t_{\text{low}} = 1.0 \text{ meV}$, $t_{\text{high}} = 0.4 \text{ meV}$, $K = 0.2 \text{ meV}$

