# 3. project

We revisit the damped harmonic oscillator in projet 2:

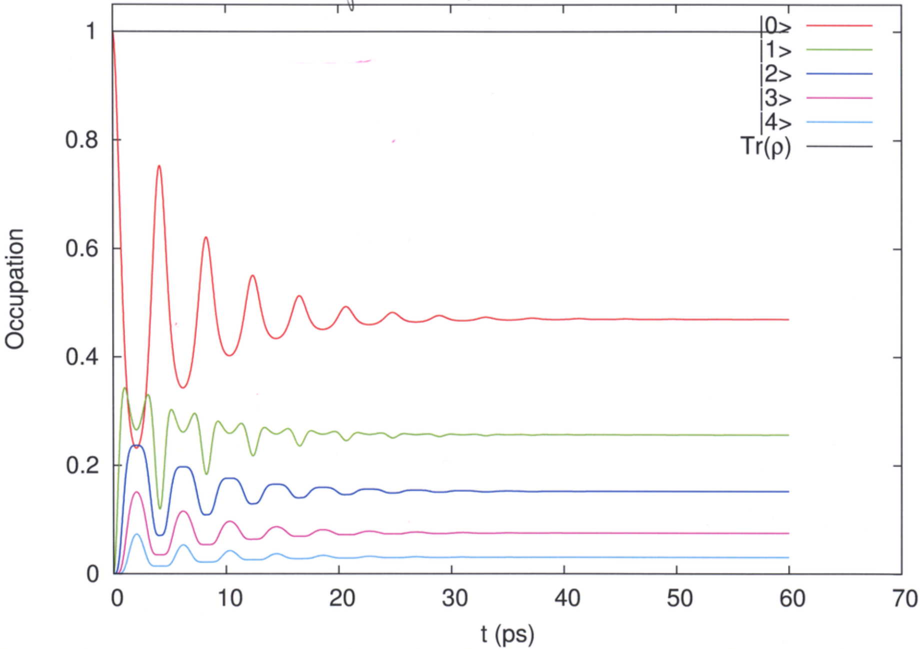
$$i\hbar\dot{\varrho} = [H,\varrho] - \frac{i}{2}\{R\delta\varrho + \delta\varrho R\} \qquad (1)$$

$$H(t) = H_0 + H'(t) \; , \; H_0 = \hbar\omega\{a^\dagger a + \tfrac{1}{2}\}$$

$$H'(t) = \hbar\Omega\{a^\dagger + a\}\,\theta(t)$$

There we solved Eq. (1) to find the time-evolution of $\varrho(t)$. With the finite damping we saw that the solution tends to a steady state as $t \to \infty$.

With dissipation, $\hbar\omega = \hbar\Omega = 1\,meV$, $N_f = 16$



The question now is if we can find the steady-state directly without doing the time-integration? ③

We can rewrite the dissipative eq. of motion (1) as

$$\dot{\rho}(t) = \Lambda[\rho(t)]$$

In the steady state $\dot{\rho} = 0$, and we need to solve

$$0 = \Lambda[\rho(t)]$$

As we have

$$i\Lambda[\rho] = \frac{1}{\hbar}[H, \rho] - \frac{i}{2\hbar}\{R\,\delta\rho + \delta\rho\,R\}$$

with

$$\delta\rho = \{\rho(t) - \rho_0\}$$

Our steady state equation can be written as

$$\boxed{A\rho + \rho B = C} \quad (2)$$

which is Lyapunov's equation, or the Sylvester equation.

Numerically, it is the best practice here to solve it directly, which can be accomplished by applying several Lapack-routines related to linear algebra transformations and generalized eigenvalue problems

But for fun let's try a different method here, that will introduce to us some new matrix manipulations worth to know.

In the Matrix Cookbook a solution is given for (2) in a N$^2$-space if the original matrices have dimension N.

$$\mathrm{vec}(\rho) = \left(I \otimes A + B^T \otimes I\right)^{-1} \mathrm{vec}(C)$$

where vectorization of matrices together with the tensorproduct of Kronecker is used.

More cleverly we have now to solve the linear equation

$$\left\{ I \otimes A + B^T \otimes I \right\} \mathrm{vec}(\rho) = \mathrm{vec}(C)$$

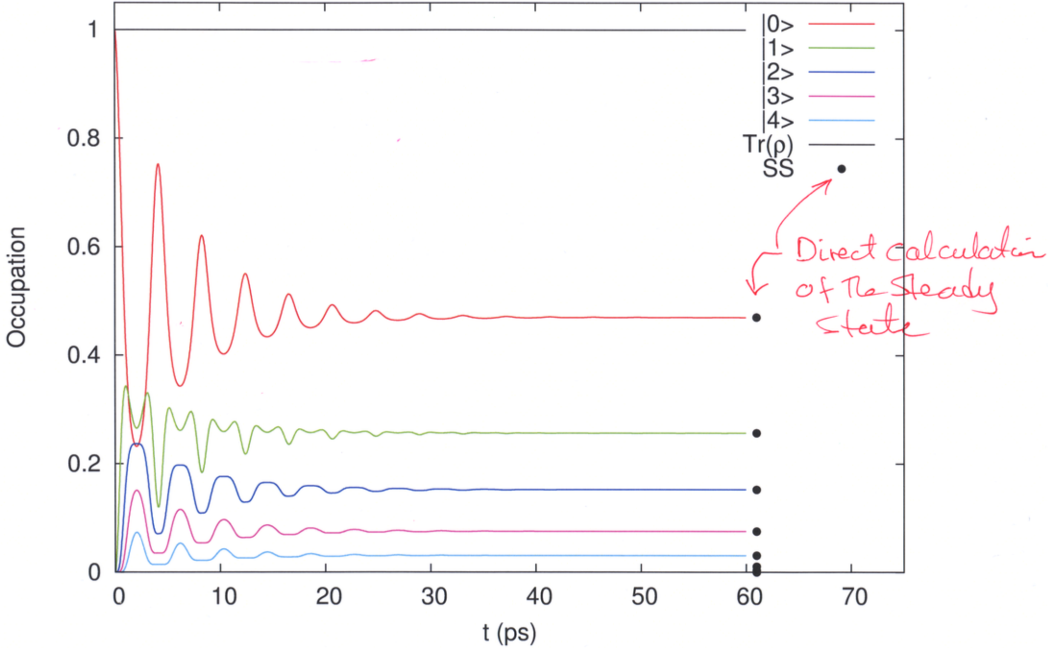
$$A = \left(H - \tfrac{i}{2} R\right), \quad B = -\left(H + \tfrac{i}{2} R\right)$$

$$C = -\tfrac{i}{2} \left\{ R \rho_0 + \rho_0 R \right\}$$

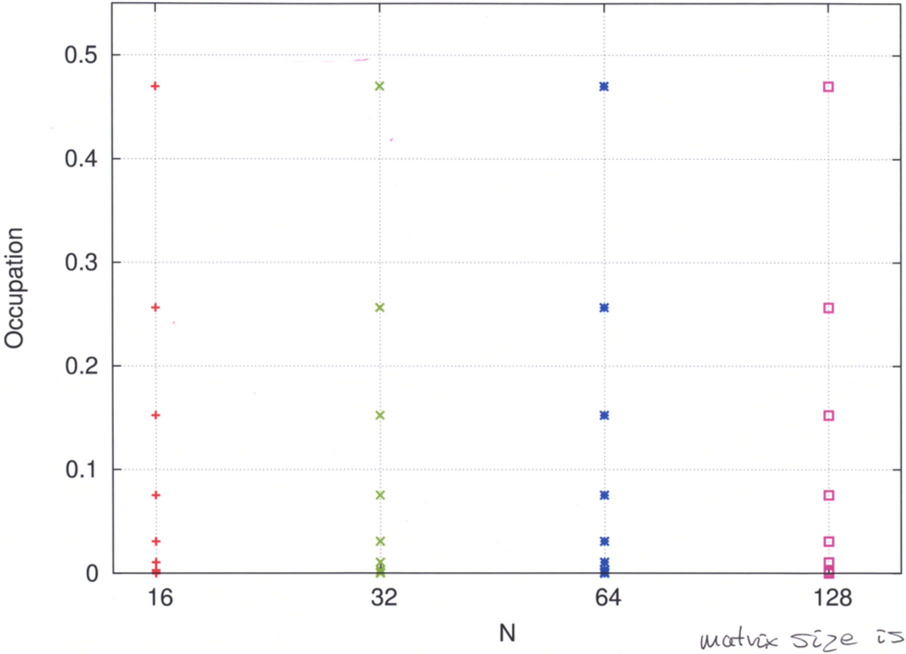
In the next page we see how the calculation for the Steady State agree.

Do this calculation, and find some interesting properties to explore further.

With dissipation, $\hbar\omega = \hbar\Omega = 1\,meV$, $N_f = 16$

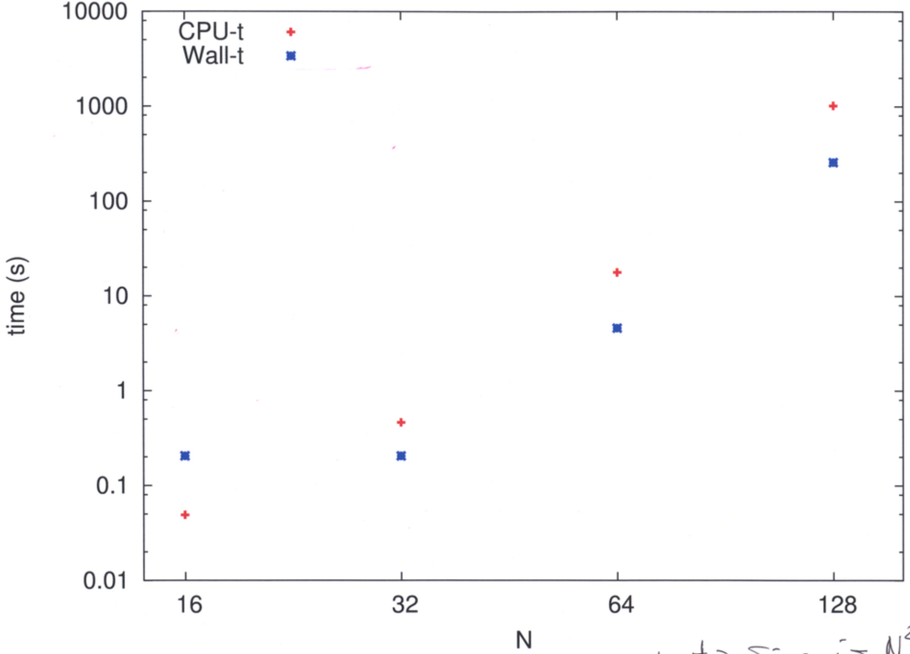


$\hbar\omega = \hbar\Omega = 1\ meV$




matrix size is $N^2 \times N^2$

$\hbar\omega = \hbar\Omega = 1\,\mathrm{meV}$




matrix size is $N^2 \times N^2$