1. project

Consider a harmonic oscillator in 1D, with

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \quad , \qquad [x,p] = i\hbar$$

introduce lowering and raising operators, such that

$$x = \frac{a}{\sqrt{2}}(a_+ + a_-) \quad , \qquad p = \frac{i\hbar}{\sqrt{2}\,a}(a_+ - a_-)$$

$$a_- |n\rangle = \sqrt{n}\,|n-1\rangle \quad , \qquad a_+ |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

on the eigenstates of H_0 $|n\rangle$ with spectrum $E_n^0 = \hbar\omega(n+\frac{1}{2})$

$a = \sqrt{\frac{\hbar}{m\omega}}$ is the natural length scale.

$$H_0 = \hbar\omega_0 \left\{ a_+ a_- + \frac{1}{2} \right\}$$

$$H_0 |n\rangle = E_n^0 |n\rangle \, , \quad E_n^0 = \hbar\omega\left(n+\frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

We can find the matrix element

$$\langle n | \frac{x}{a} | m \rangle = \frac{1}{2}\sqrt{n+m+1}\, \delta_{|n-m|,1}$$

So, in the basis $\{|n\rangle\}$ the operator x can be represented by an infinite matrix. In the same way H_0 is represented by an infinite diagonal matrix

$$\langle n | H_0 | m \rangle = \hbar\omega\left(n+\frac{1}{2}\right)\delta_{n,m}$$

How about x^2?

$$\langle n|\left(\frac{x}{a}\right)^2|m\rangle = \langle n|\left(\frac{x}{a}\right)\left(\frac{x}{a}\right)|m\rangle$$

Now use the completeness relation

$$\mathbb{1} = \sum_{p=0}^{\infty} |p\rangle\langle p|$$

$$\langle n|\left(\frac{x}{a}\right)^2|m\rangle = \sum_{p=0}^{\infty} \langle n|\left(\frac{x}{a}\right)|p\rangle\langle p|\left(\frac{x}{a}\right)|m\rangle$$

If we had noted the matrix for the operator by $\mathbb{X}$ then we see from this that the matrix for x^2 is just $\mathbb{X}\cdot\mathbb{X}$ where matrix multiplication is used.

a) find the energy spectrum for

$$H = H_0 + \lambda \hbar\omega \left(\frac{x}{a}\right)^4$$

for $\lambda \in [0, 1.2]$ for an appropriately truncated basis. test the truncation.

b) The eigen states $|\alpha)$ of H are a linear combination of the original basis

$$|\alpha) = \sum_{n=0}^{\infty} C_{\alpha n} |n\rangle$$

and the coefficients $C_{\alpha n}$ come from the eigenvectors of H. Show with column-graphs the contribution of basis states to the 3 lowest energy states $|\alpha)$. $|C_{\alpha n}|^2$ v.s. n

c) More complex functions of x

There is a unitary transformation U that can get X on a diagonal form

$$X = U X_d U^+ = U \operatorname{diag}\{\overbrace{\lambda_1 \cdots \lambda_N}^{\text{eigenvalues of } X}\} U^+$$

$$\rightarrow f(X) = U \operatorname{diag}\{f(\lambda_1) \cdots f(\lambda_N)\} U^+$$

where λ_i is the i^{th} eigenvalue of X and U is the unitary matrix formed of it's eigenvectors.

We can scale

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

using a, and for fun we can also use $p = \hbar k$ such that the dimension of k: $[k] \sim \frac{1}{L}$

$$H_0 = \frac{(\hbar k a)^2}{2ma^2} + \frac{a^2}{2} m\omega^2 \left(\frac{x^2}{a^2}\right) = \hbar\omega \left\{ \frac{(ka)^2}{2} + \frac{1}{2}\left(\frac{x^2}{a}\right) \right\}$$

The potential of the harmonic oscillator is thus

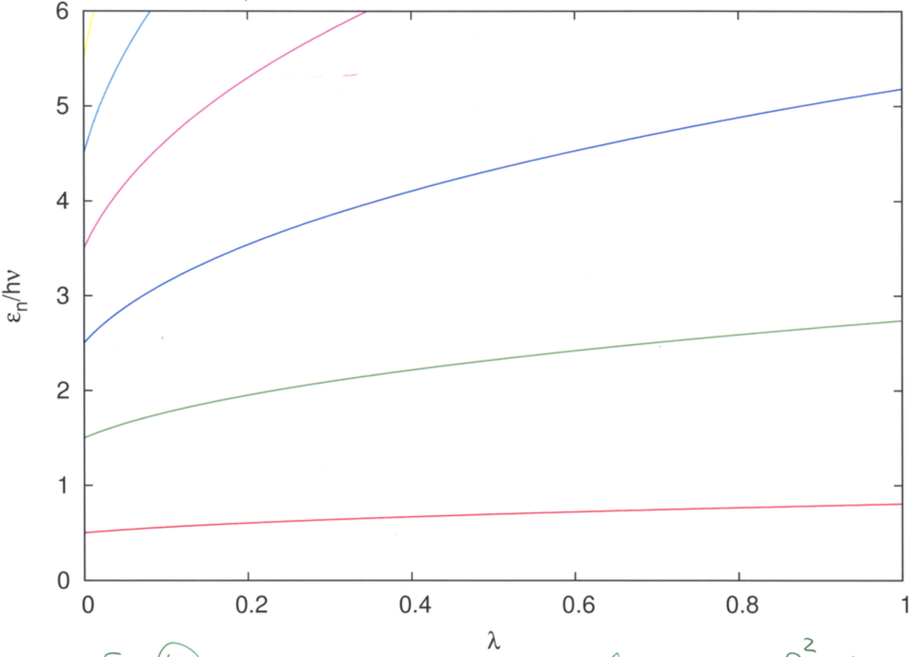
$$V_0(x) = \frac{1}{2}\hbar\omega \left(\frac{x}{a}\right)^2$$

Can we use the method to find the spectrum for

$$V(x) = \hbar\omega \left[\frac{1}{2}\left(\frac{x}{a}\right) \cdot \tanh\left(\frac{x}{a}\right) - \frac{1}{2}\left(\frac{x}{a}\right)^2 \right]$$

Hwo large basis do we need to get a reasonable accuracy for the 10 lowest states in $V(x)$?
Check $V(x)$ in gnuplot, notice that by an appropriate coefficient in the argument of tanh we are really changing the parabolic potential into almost a V-shaped one.

Orkuróf $\quad H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda\hbar\omega\left(\frac{x}{a}\right)^4$ $\quad$ 2f



Fundið í grunni 128 eiginfalla $\quad H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$