(1)

Sphere with $V_0(\theta) = k\cos(3\theta)$, no charge specified.

$$\nabla^2 V = 0 \quad \text{or} \quad \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial R}\right) = 0$$

Assume $V(R,\theta) = \Gamma(R)\Theta(\theta)$ separability

General solutions with no singularity in angular variables

$$V_n(R,\theta) = \left\{A_n R^n + B_n R^{-(n+1)}\right\} P_n(\cos\theta)$$

Inside, $R < a$

No charge $\rightarrow$ $V_n^{\text{I}}(R,\theta) = A_n R^n P_n(\cos\theta)$

(2)

Outside, $R > a$

$V \xrightarrow[R\to\infty]{} 0$ $\rightarrow$ $V_n^{\text{II}}(R,\theta) = B_n R^{-(n+1)} P_n(\cos\theta)$

B.C at $R = a$

First, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$, $P_1(x) = x$, $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$

$\rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$

So $V_0(\theta) = k\cos(3\theta) = \frac{k}{5}\left\{8P_3(\cos\theta) - 3P_1(\cos\theta)\right\}$

The solution inside and outside the sphere will thus have these two components, only.
The are orthogonal and can be matched individually

(3)

I $\quad A_3 R^3 P_3(\cos\theta) + A_1 R P_1(\cos\theta)$

II $\quad B_3 R^{-4} P_3(\cos\theta) + B_1 R^{-2} P_1(\cos\theta)$

At $R = a$ "$V^{\text{I}}(a) = V_0 = V^{\text{II}}(a)$"

$A_3 a^3 = \frac{k}{5}8 \qquad \frac{B_3}{a^4} = \frac{k}{5}8$

$A_1 a = -\frac{k}{5}3 \qquad \frac{B_1}{a^2} = -\frac{k}{5}3$

$\rightarrow A_3 = \frac{8k}{5a^3} \qquad B_3 = \frac{2ka^4}{5}$

$A_1 = -\frac{3k}{5a} \qquad B_1 = -\frac{3ka^2}{5}$

(4)

The solution is then

I: $\quad V^{\text{I}}(R,\theta) = \frac{k}{5}\left\{8\left(\frac{R}{a}\right)^3 P_3(\cos\theta) - 3\left(\frac{R}{a}\right)P_1(\cos\theta)\right\}$

II: $\quad V^{\text{II}}(R,\theta) = \frac{k}{5}\left\{8\left(\frac{a}{R}\right)^4 P_3(\cos\theta) - 3\left(\frac{a}{R}\right)^2 P_1(\cos\theta)\right\}$

The potential has both dipole and quadrupole moments.

To calculate the surface density of charge we need $\overline{E}$.

$$\overline{E} = -\overline{\nabla}V(R,\theta) = -\hat{a}_R\partial_R V(R,\theta) - \hat{a}_\theta\frac{1}{R}\partial_\theta V(R,\theta)$$

and only the normal $\Leftrightarrow$ radial part

$$E_R = -\partial_R V(R,\theta)$$

remember (as $\epsilon^{I} = \epsilon^{II} = \epsilon_0$)

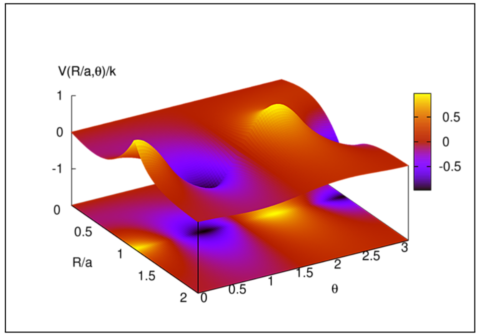
$$\epsilon_0 \hat{a}_R \cdot \left(\bar{E}^{II}(a) - \bar{E}^{I}(a) \right) = \rho_s(a,\theta)$$

$$\rightarrow \rho_s(\theta) = \epsilon_0 \left(E_R^{II}(a,\theta) - E^{I}(a,\theta) \right)$$

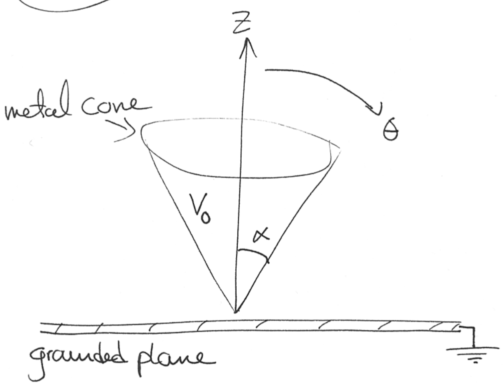
$$= - \epsilon_0 \partial_R V^{II}(R,\theta) \Big|_{R=a^+} + \epsilon_0 \partial_R V^{I}(R,\theta) \Big|_{R=a^-}$$

$$= \frac{\epsilon_0 k}{5a} \left\{ 32 P_3(\cos\theta) - 6 P_1(\cos\theta) \right\} + \frac{\epsilon_0 k}{5a} \left\{ 24 P_3(\cos\theta) - 3 P_1(\cos$$

$$= \frac{\epsilon_0 k}{5a} \left\{ 56 P_3(\cos\theta) - 9 P_1(\cos\theta) \right\}$$



P4-27



a) Calculate $V(\theta)$ outside the cone, between the cone and the plane

The plane is homogeneously grounded to infinity and the cone is everywhere at V_0

The only possible variable V depends on is θ in spherical coordinates

$$\nabla^2 V(\theta) \iff \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} V\theta \right) = 0$$

or

$$\frac{d}{d\theta} \left(\sin\theta \frac{dV}{d\theta} \right) = 0$$

Integrate indefinitely

Constant of integration

$$\rightarrow \sin\theta \frac{dV}{d\theta} = C_1$$

$$\rightarrow \frac{dV}{d\theta} = \frac{C_1}{\sin\theta}$$

use Maxima or GR 2.515.2

$$\rightarrow V(\theta) = C_1 \ln\left\{ \tan\left(\frac{\theta}{2}\right) \right\} + C_2$$

Now use the boundary conditions

$$V(\alpha) = V_0 \qquad C_1 \ln\left\{\tan\left(\tfrac{\alpha}{2}\right)\right\} + C_2 = V_0$$

$$V(\pi/2) = 0 \qquad C_1 \ln\{\underbrace{\tan\left(\tfrac{\pi}{4}\right)}_{=1}\} + C_2 = 0$$

$$\rightarrow C_2 = 0$$

and $C_1 \ln\left\{\tan\left(\frac{\alpha}{2}\right)\right\} = V_0 \quad \rightarrow C_1 = \dfrac{V_0}{\ln\left\{\tan\left(\frac{\alpha}{2}\right)\right\}}$

and thus the solution is

$$V(\theta) = V_0 \frac{\ln\left\{\tan\left(\frac{\theta}{2}\right)\right\}}{\ln\left\{\tan\left(\frac{\alpha}{2}\right)\right\}}$$

b) $|E|$ in the same region

Spherical coordinates

$$\rightarrow \bar{E} = -\hat{a}_\theta \frac{dV}{R\,d\theta} = -\frac{\hat{a}_\theta}{R} \frac{V_0}{\ln\left\{\tan\left(\frac{\alpha}{2}\right)\right\}} \frac{1}{2} \frac{1}{\tan\left(\frac{\theta}{2}\right)} \frac{1}{\cos^2\left(\frac{\theta}{2}\right)}$$

$$= -\hat{a}_\theta \frac{V_0}{R \ln\left\{\tan\left(\frac{\alpha}{2}\right)\right\} \sin\theta}$$

$\bar{E}$ depends on R !

c) $\bar{E}$ is only perpendicular to the cone and the plane, due to $\hat{a}_\theta$!

Find ρ_s at these surfaces

On the cone

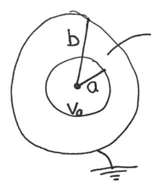
$$\rho_s(R) = \epsilon_0 E(\alpha) = -\frac{\epsilon_0 V_0}{R \ln\left\{\tan\left(\frac{\alpha}{2}\right)\right\} \sin\alpha}$$

on the plane

$$\rho_s(R) = -\epsilon_0 E\left(\frac{\pi}{2}\right) = \frac{\epsilon_0 V_0}{R \ln\left\{\tan\left(\frac{\alpha}{2}\right)\right\}}$$

Both surfaces have a charge distribution that weakens as $\frac{1}{R}$ away from their contact point.
The charges accumulate where the force between them is the largest

Concentric conducting spheres



$\sigma = \frac{m}{r} + k$

a) Find R. Not a homogeneous σ, but a radial symmetry

$\bar{J} = \frac{I}{4\pi r^2} \hat{a}_r$ homogeneous current density

$$\bar{J} = \sigma \bar{E} \rightarrow \bar{E} = \frac{\bar{J}}{\sigma} = \frac{I}{4\pi r^2} \frac{r}{m + kr} \hat{a}_r$$

So we can find the potential difference

$$V_0 = -\int_b^a \bar{E} \cdot d\bar{l} = -\int_b^a \frac{I\,dr}{4\pi r(m + kr)}$$

$$V_0 = -\frac{I}{4\pi}\int_b^a \frac{dr}{rm + kr^2} = -\frac{I}{4\pi k}\int_b^a \frac{dr}{r^2 + \left(\frac{m}{k}\right)r}$$

$$= \frac{I}{4\pi k}\,\frac{k}{m}\left\{-\ln(m+bk) + \ln(m+ak) + \ln(b) - \ln(a)\right\}$$

$$= \frac{I}{4\pi m}\left[\ln\left(\frac{m+ak}{m+bk}\right) + \ln\left(\frac{b}{a}\right)\right]$$

$$\rightarrow R = \frac{V_0}{I} = \frac{1}{4\pi m}\left\{\ln\left(\frac{m+ak}{m+bk}\right) + \ln\left(\frac{b}{a}\right)\right\}$$

b) find the surface charge on each surface

Metal surfaces, so only E on one side

$$\bar{E} = \frac{I}{4\pi}\,\frac{\hat{a}_r}{kr^2 + mr}$$

Inner surface $r = a$

$$\rho_s(a) = \epsilon_0\,\bar{E}(a)\cdot\hat{a}_r = \frac{\epsilon_0 I}{4\pi}\,\frac{1}{ka^2 + ma}$$

Outer surface $r = b$

$$\rho_s(b) = -\epsilon_0\,\bar{E}(b)\cdot\hat{a}_r = -\frac{\epsilon_0 I}{4\pi}\,\frac{1}{kb^2 + mb}$$

c) Volume charge density between the spheres, $a < r < b$

$$\rho(r) = \bar{\nabla}\cdot\bar{D} = \frac{\partial}{\partial r}(\epsilon_0 E_r)$$

$$E_r = \frac{I}{4\pi(kr^2 + mr)}$$

$$\rightarrow \rho(r) = \frac{\partial}{\partial r}\left(\epsilon_0\frac{I}{4\pi(kr^2+mr)}\right) = \frac{\epsilon_0 I}{4\pi k}\,\frac{\partial}{\partial r}\left(\frac{1}{r^2 + \left(\frac{m}{k}\right)r}\right)$$

$$= -\frac{\epsilon_0 I}{4\pi k}\,\frac{2r + \left(\frac{m}{k}\right)}{\left(r^2 + \left(\frac{m}{k}\right)r\right)^2}$$

d) We know V_0 and σ, but not I initially.

We could use a) to express I in terms of known quantities

$$I = \frac{V_0}{R(a,b,m,k)} = \frac{V_0}{\frac{1}{4\pi m}\left\{\ln\left(\frac{m+ak}{m+bk}\right) + \ln\left(\frac{b}{a}\right)\right\}}$$

then we have total I, and also the current density

$$\bar{J} = \frac{I}{4\pi r^2}\,\hat{a}_r$$

e) $\lim_{m\to 0} R$?

$$\frac{1}{4\pi m}\left\{\ln\left(\frac{m+ak}{m+bk}\right) + \ln\left(\frac{b}{a}\right)\right\} \sim \frac{1}{4\pi m}\left\{\frac{(bk-ak)m}{abk^2} + o(m^2)\right\}$$

6

$$= \frac{1}{4\pi} \frac{(b-a)}{abk} + o(m^2)$$

and as expected $R \to 0$ if $a \to b$

$$\lim_{m \to 0} R = \frac{1}{4\pi} \frac{b-a}{abk}$$

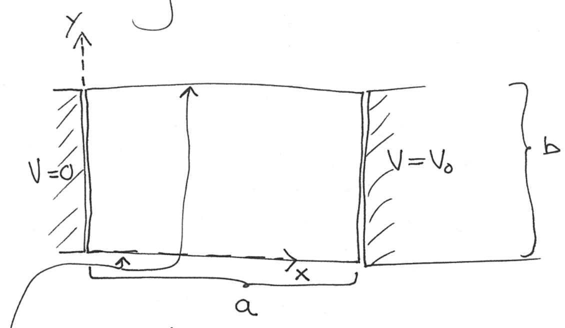
7

P.5-22

Rectangular sheet with σ ← homogeneous, so we can use

$$\bar{J} = \sigma \bar{E}$$

and both $\bar{J}$ and $\bar{E}$ fullfil similar equations



Here $\bar{J} \cdot \hat{a}_n = 0$ (and thus also $\bar{E} \cdot \hat{a}_n = 0$)

$\nabla^2 V(x,y) = 0$, use separability

for symmetry reasons $V(y) = C_1$ constant.

Then we also fullfil the B.C. at $y=0$ and $y=b$

8

$V(y)$-solution means $k_y = 0$, thus we also need $k_x = 0$ since $k_y^2 + k_x^2 = 0$

$$\to V(x) = Ax + B$$

B.C. at $x=0$ and $x=a$ give $B=0$ and $A = \frac{V_0}{a}$

$$\to V(x,y) = \frac{V_0}{a} x \quad (\text{as } C_1 = 1)$$

b) $\bar{E} = -\bar{\nabla} V \to \bar{E} = -\hat{a}_x \left(\frac{V_0}{a}\right)$

and thus $\bar{J} = -\hat{a}_x \left(\frac{\sigma V_0}{a}\right)$

9

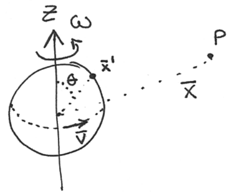
If we calculate the surface charge densities for the contacts here we would get

$$\rho_s(0) = -\epsilon_0 \frac{V_0}{a} \quad \text{and} \quad \rho_s(a) = \epsilon_0 \frac{V_0}{a}$$

but do we really have surface or line contacts? Dimensional analysis gives $\left[\epsilon_0 \frac{V_0}{a}\right] \sim \frac{Q}{L^2}$ as should be. We are implicitly implying with the Laplace equation ($k_z = 0$) a homogeneous solution in the z-direction, but the condition $\rho_s = \rho_\ell \delta(z)$ to introduce a line charge at the contact would not support that. This is a common problem in reduced dimensionality.

1

Thin spherical shell of radius a and surface charge density ρ_s



The velocity of each point $\bar{x}'$ on the sphere is

$$\bar{v}(\bar{x}') = \omega\, \hat{a}_z \times \bar{x}'$$

giving the more familiar speed

$$v = \omega a \sin\theta$$

The velocity $\bar{v}$ is alwas parallel to $\hat{a}_\varphi$

The surface current density is

$$\bar{J}_s(\bar{x}') = \rho_s\, \bar{v}(\bar{x}') = \rho_s \omega\, \hat{a}_z \times \bar{x}'$$

2

One can thus write the 3D current density

$$\bar{J}(\bar{x}') = \bar{J}_s(\bar{x}')\,\delta(|\bar{x}'| - a)$$

and use directly (6-23)

$$\bar{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\bar{J}}{R}\, d\bar{v}' \qquad (*)$$

Or building an equation for $\bar{J}_s$ like (6-27) was derived from (6-23) for a line current.

Anyway we need to fully write $(*)$ as

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\int \frac{\bar{J}(\bar{x}')}{|\bar{x} - \bar{x}'|}\, d^3x'$$

3

to remind us of the test ahead. Then

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\int \frac{\bar{J}_s(\bar{x}')\,\delta(|\bar{x}'| - a)}{|\bar{x} - \bar{x}'|}(x')^2 d\Omega'$$

$$= \frac{\mu_0}{4\pi}\rho_s a^2 \omega\, \hat{a}_z \times \int \frac{\bar{x}'}{|\bar{x} - \bar{x}'|}\, d\Omega' \qquad \underline{\text{with } |\bar{x}'| = a}$$

and $|\bar{x}| = r$ is the distance of the observer at P from the center of the sphere

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\rho_s a^2 \omega\, \hat{a}_z \times \bar{F}(\bar{x})$$

with

$$\bar{F}(\bar{x}) = \int \frac{\bar{x}'\, d\Omega'}{|\bar{x} - \bar{x}'|}$$

4

$\bar{F}(\bar{x})$ is a vector quantity. After the $d\Omega'$ integration we can guess it can only be proportional to $\bar{x}$

$$\bar{F}(\bar{x}) = F(r)\,\bar{x}$$

Then

$$\bar{x}\cdot\bar{F}(\bar{x}) = r^2 F(r) = \int \frac{\bar{x}\cdot\bar{x}'\, d\Omega'}{|\bar{x} - \bar{x}'|}$$

We parametrize this with β - the angle between $\bar{x}$ and $\bar{x}'$ $\longrightarrow$ $\bar{x}\cdot\bar{x}' = ra\cos\beta$

$$\longrightarrow F(r) = \frac{1}{r^2}\int \frac{\bar{x}\cdot\bar{x}'\, d\Omega'}{|\bar{x} - \bar{x}'|}$$

and

$$d\Omega' = 2\pi \sin\beta\, d\beta$$

5

$$F(r) = \frac{a}{r}\int_0^{\pi} \frac{\cos\beta \cdot 2\pi \cdot \sin\beta\, d\beta}{\sqrt{r^2+a^2-2ra\cos\beta}}$$

Change variable $u = \cos\beta$

$$\rightarrow \quad F(r) = 2\pi\frac{a}{r}\int_{-1}^{1}\frac{u\,du}{\sqrt{r^2+a^2-2rau}}$$

$$= 2\pi\frac{a}{r}\begin{cases} \dfrac{1}{r}\displaystyle\int_{-1}^{1}\frac{u\,du}{\sqrt{1+\left(\frac{a}{r}\right)^2-2\left(\frac{a}{r}\right)u}} & \text{if } r<a \\[3ex] \dfrac{1}{a}\displaystyle\int_{-1}^{1}\frac{u\,du}{\sqrt{1+\left(\frac{r}{a}\right)^2-2\left(\frac{r}{a}\right)u}} & \text{if } r>a \end{cases}$$

(use GR 2.222.2 with care)

6

$$F(r) = 2\pi\frac{a}{r}\begin{cases} \dfrac{2r}{3a^2} \longrightarrow \dfrac{4\pi}{3}\dfrac{1}{a} & \text{if } r<a \\[2ex] \dfrac{2a}{3r^2} \longrightarrow \dfrac{4\pi}{3}\dfrac{a^2}{r^3} & \text{if } r>a \end{cases}$$

and thus

$$\bar{A}(\bar{x}) = \begin{cases} \dfrac{\mu_0\rho_s a}{3}\,\omega\left(\hat{a}_z\times\bar{x}\right) & \text{if } r<a \\[3ex] \dfrac{\mu_0\rho_s a^4}{3r^3}\,\omega\left(\hat{a}_z\times\bar{x}\right) & \text{if } r>a \end{cases}$$

6b

A much better method for the integration is using

$$\frac{1}{|\bar{x}-\bar{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\beta) \qquad \begin{array}{l} r_< = \min(r,a) \\ r_> = \max(r,a)\end{array}$$

and then

$$\int\frac{\cos\beta}{|\bar{x}-\bar{x}'|}\,d\Omega' = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\underbrace{\int P_1(\cos\beta)P_l(\cos\beta)\,d\Omega'}_{\text{use orthogonality}}$$

$$= \frac{4\pi}{3}\frac{r_<}{r_>^2}$$ and so on . . . .

7

$r<a$

$$\bar{B} = \bar{\nabla}\times\bar{A} = \frac{\mu_0\rho_s a\omega}{3}\,\bar{\nabla}\times\left(\hat{a}_z\times\bar{x}\right)$$

$$= \frac{\mu_0\rho_s a\omega}{3}\left\{\hat{a}_z\overbrace{(\bar{\nabla}\cdot\bar{x})}^{3} - \cancel{\bar{x}(\bar{\nabla}\cdot\hat{a}_z)} + \cancel{(\bar{x}\cdot\bar{\nabla})\hat{a}_z} - \overbrace{(\hat{a}_z\cdot\bar{\nabla})\bar{x}}^{\hat{a}_z}\right\}$$

$$= \frac{2\mu_0\rho_s a\omega}{3}\,\hat{a}_z \qquad \text{a constant}$$

$r>a$

$$\bar{B} = \bar{\nabla}\times\bar{A} = \frac{\mu_0\rho_s a^4\omega}{3}\,\bar{\nabla}\times\left(\frac{\hat{a}_z\times\bar{x}}{r^3}\right)$$

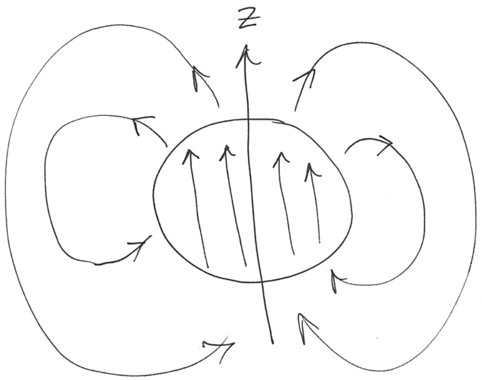
$$= \frac{\mu_0\rho_s a^4\omega}{3}\left\{\bar{\nabla}\left(\frac{1}{r^3}\right)\times\left(\hat{a}_z\times\bar{x}\right) + \frac{1}{r^3}\bar{\nabla}\times\left(\hat{a}_z\times\bar{x}\right)\right\}$$

8

$$\bar{B} = \frac{\mu_0 \rho_s a^4 \omega}{3} \frac{3\hat{x}(\hat{a}_z \cdot \hat{x}) - \hat{a}_z}{r^3} \qquad r > a$$

which is a dipole field of a dipole with strength

$$\bar{m} = \frac{4\pi}{3} \rho_s a^4 \omega \, \hat{a}_z$$



use $\quad \bar{\nabla}\left(\frac{1}{r^3}\right) = -\frac{3\bar{x}}{r^5}$

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$$

also valuable are

$\bar{\nabla}\cdot\bar{x} = 3 \qquad \bar{\nabla}\times\bar{x} = 0$

$\bar{\nabla}\cdot\bar{n} = \frac{2}{r} \qquad \bar{\nabla}\times\bar{n} = 0$

if $\quad r = |\bar{x}|$ and $\bar{n} = \frac{\bar{x}}{r}$

P6-26 9

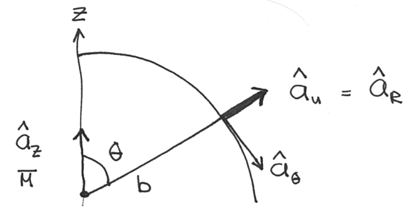
Ferromagnetic sphere of radius b with magnetization $\bar{M} = \hat{a}_z M_0$

a) find equivalent magnetization current densities $\bar{J}_m$ and $\bar{J}_{ms}$

$\bar{J}_m = \bar{\nabla} \times \bar{M}$ and $\bar{J}_{ms} = \bar{M} \times \hat{a}_n$

$$\bar{J}_m = \bar{\nabla} \times \bar{M} = M_0 \bar{\nabla} \times \hat{a}_z = 0$$

$\bar{J}_{ms} = M_0 \, \hat{a}_z \times \hat{a}_n$, here $\hat{a}_n = \hat{a}_R$



we need $\hat{a}_z$ in spherical coordinates

$$\hat{a}_z = \hat{a}_R \cos\theta - \hat{a}_\theta \sin\theta$$

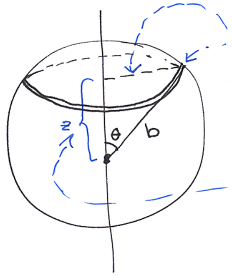
10

$$\rightarrow \bar{J}_{ms} = M_0 \, \hat{a}_z \times \hat{a}_n = M_0 \left(\hat{a}_R \cos\theta - \hat{a}_\theta \sin\theta\right) \times \hat{a}_R$$

$$= \hat{a}_\varphi M_0 \sin\theta$$

b) $\bar{B}$ at the center of the sphere?

We have the current loop $\bar{J}_{ms}(\theta)$ with radius $b\sin\theta$ and want to observe $\bar{B}$ at the height z



Use (6-38)

$$d\bar{B} = \hat{a}_z \frac{\mu_0 (b\sin\theta)^2 (J_{ms} \cdot b\,d\theta)}{2\left\{(b\cos\theta)^2 + (b\sin\theta)^2\right\}^{3/2}}$$

11

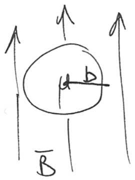
$$\rightarrow d\bar{B} = \hat{a}_z \frac{\mu_0 M_0}{2} \sin^3\theta$$

Now we sum up all contributions for θ from 0 to π

$$\bar{B} = \hat{a}_z \frac{\mu_0 M_0}{2} \int_0^{\pi} \sin^3\theta \, d\theta = \hat{a}_z \frac{2\mu_0 M_0}{3} = \frac{2}{3}\mu_0 \bar{M}$$

1

Sphere with constant μ in an external $\bar{B} = B_0 \hat{a}_z$



No free currents $\rightarrow$ $\bar{\nabla} \times \bar{H} = 0$

and if $\bar{H} = -\bar{\nabla}\phi_m$ then this is satisfied

and as $\bar{\nabla}\cdot\bar{B} = 0$ we have

$$\bar{\nabla}\cdot\bar{B} = \bar{\nabla}\cdot(\mu\bar{H}) = 0 \quad \text{or} \quad \nabla^2\phi_m = 0$$

In order to give $\bar{B} = B_0\hat{a}_z$ far away from the sphere we must have

$$\phi_m^{o}(\bar{x}) = -\frac{B_0}{\mu_0} z = -\frac{B_0}{\mu_0} R\cos\theta$$

indicating an external field

which we will use as one of the boundary conditions

2

We have earlier solved the Laplace equation for an azimuthal symmetric problem in spherical coordin. The general solution is

$$\phi_m(R,\theta) = \sum_{n=0}^{\infty}\left\{A_n R^n + B_n R^{-(n+1)}\right\} P_n(\cos\theta)$$

The external asymptotic solution requires that only terms with P_1 can be non vanishing

Inside

$$\phi_m^{i}(R,\theta) = A_1 R\cos\theta, \qquad R<b$$

Outside

$$\phi_m^{o}(R,\theta) = -\frac{B_0}{\mu_0} R\cos\theta + B_1\frac{\cos\theta}{R^2}, \qquad R>b$$

3

The general boundary condition for the tangential component is

$$\hat{a}_{n2}\times(\bar{H}_1 - \bar{H}_2) = \bar{J}_s.$$

We have no free currents and in our case

$$\hat{a}_R \times \left(\bar{H}_\theta^{o} - \bar{H}_\theta^{i}\right) = 0$$

as we have no φ-component. Rewritten for the magnetic scaler potential ϕ_m this is

$$-\frac{1}{b}\left(\frac{\partial\phi_m^{i}}{\partial\theta}\right)_{R=b} = -\frac{1}{b}\left(\frac{\partial\phi_m^{o}}{\partial\theta}\right)_{R=b}$$

giving

$$-\frac{1}{b}A_1 b = -\frac{1}{b}(-B_0)\frac{b}{\mu_0} - \frac{B_1}{b}\frac{1}{b^2}$$

4

or

$$b^3 A_1 - B_1 = -\frac{b^3 B_0}{\mu_0}$$

The other boundary condition is

$$B_{1n} = B_{2n} \qquad (\mu_1 H_{1n} = \mu_2 H_{2n})$$

which in our case is

$$-\mu\left(\frac{\partial\phi_m^{i}}{\partial R}\right)_{R=b} = -\mu_0\left(\frac{\partial\phi_m^{o}}{\partial R}\right)_{R=b}$$

or

$$-\mu A_1 = \mu_0\left(\frac{B_0}{\mu_0} + 2B_1\frac{1}{b^3}\right)$$

and thus

$$\frac{\mu}{\mu_0} b^3 A_1 + 2B_1 = -b^3\frac{B_0}{\mu_0}$$

(5)

So we have to solve

$$b^3 A_1 - B_1 = -\frac{b^3 B_0}{\mu_0}$$

$$\frac{\mu}{\mu_0} b^3 A_1 + 2B_1 = -\frac{b^3 B_0}{\mu_0}$$

$$\hookrightarrow \quad A_1 = -\frac{3B_0}{\left(\frac{\mu}{\mu_0}+2\right)\mu_0}, \qquad B_1 = \frac{\left(\frac{\mu}{\mu_0}-1\right)B_0 b^3}{\left(\frac{\mu}{\mu_0}+2\right)\mu_0}$$

So finally for ϕ_m we have

$$\boxed{\begin{aligned} \phi_m^i(R,\theta) &= -\frac{3B_0}{\left(\frac{\mu}{\mu_0}+2\right)\mu_0} R\cos\theta \\ \phi_m^o(R,\theta) &= -\frac{B_0}{\mu_0} R\cos\theta + \frac{\left(\frac{\mu}{\mu_0}-1\right)B_0 b^3}{\left(\frac{\mu}{\mu_0}+2\right)\mu_0} \frac{\cos\theta}{R^2} \end{aligned}}$$

(6)

No surprize is that $\phi_m(R,\theta)$ is continous at $R=b$ due to the continuity condition on B_n at the boundary.

<u>inside</u>

$$\overline{H}^i = -\overline{\nabla}\phi_m^i = \frac{3B_0}{\left(\frac{\mu}{\mu_0}+2\right)\mu_0}\hat{a}_z$$

$$\overline{B}^i = \mu\overline{H}^i = \frac{3B_0\frac{\mu}{\mu_0}}{\left(\frac{\mu}{\mu_0}+2\right)}\hat{a}_z$$

$$\overline{M} = \chi_m \overline{H}^i = \frac{3\left(\frac{\mu}{\mu_0}-1\right)B_0}{\left(\frac{\mu}{\mu_0}+2\right)\mu_0}\hat{a}_z$$

with $\chi_m = \frac{\mu}{\mu_0} - 1 = \mu_r - 1$

(7)

<u>Outside</u>

$$\overline{H}^o = \frac{B_0}{\mu_0}\hat{a}_z + \left(\frac{\mu_r-1}{\mu_r+2}\right)\left(\frac{b^3}{R^3}\right)\frac{B_0}{\mu_0}\left(2\hat{a}_R\cos\theta + \hat{a}_\theta\sin\theta\right)$$

$$\overline{B}^o = \mu_0\overline{H}^i$$

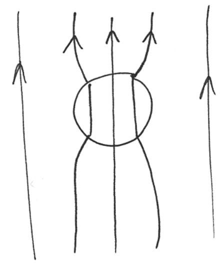
$$\overline{M}^o = 0$$

Inside the sphere all fields, $\overline{B}, \overline{H}, \overline{M}$ are uniform position independent and point in the $\hat{a}_z$ direction.

<u>Non-Magnetic</u> $\mu_r = 1 = \left(\frac{\mu}{\mu_0}\right) \;\rightarrow\; \overline{B}^i = B_0\hat{a}_z$, $\overline{M} = 0$

<u>Paramagnetic</u> $\mu_r > 1 \;\rightarrow$ <u>denser</u> field lines inside the sphere

(8)



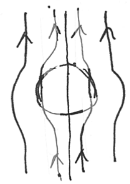
paramagnetic

for strong <u>ferromagnetic</u> material

$\mu_r \rightarrow \infty$

and $\overline{B}^i \rightarrow 3B_0\hat{a}_z$

<u>Diamagnetic</u> $\mu_r < 1$ field lines are expelled

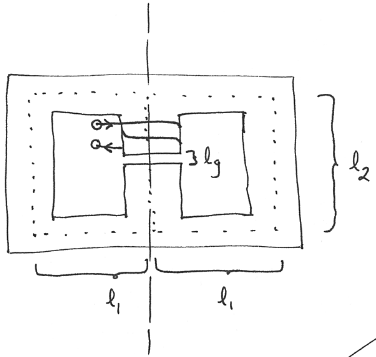


<u>perfect diamagnet</u> $\mu_r \rightarrow 0$

all lines are repelled

$\overline{B}^i = 0$, $\overline{H}^i = \frac{3B_0}{2\mu_0}\hat{a}_z$, $\overline{M}^i = -\frac{3B_0}{2\mu_0}\hat{a}_z$

6-28 (9)



$N = 200$, $I = 3A$

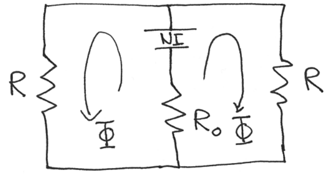
$\mu_r = \frac{\mu}{\mu_0} = 5000$

$S = 10^{-3}\,m^2$, $l_g = 0{,}002m$

$l_1 = 0{,}2m$, $l_2 = 0{,}24\,m$

Equivalent curcuit

Symmetry (mirror around the center core)



Flux in center core $\Phi_0 = 2\Phi$ wher Φ is the flux in the "3/4" what is left of each loop

R_0: Reluctance of the center core

R: -11- of the rest "$\frac{3}{4}$" of a loop

(10)

Using Example 6-10

$$R_0 = \frac{l_g}{\mu_0 S} + \frac{l_2 - l_g}{\mu S}$$

$$R = \frac{l_2 + 2l_1}{\mu S}$$

a)

$$\Phi_0 R_0 + \Phi R = NI$$

$$\Phi_0 \left(R_0 + \frac{R}{2}\right) = NI$$

$$\rightarrow \quad \Phi_0 = \frac{NI}{R_0 + R/2}$$

(11)

b)

$$H = \frac{\Phi}{\mu S} \quad \text{in } R$$

$$\rightarrow \quad H_0 = \frac{2\Phi}{\mu S} \quad \text{in central core}$$

$$H_g = \frac{2\Phi}{\mu_0 S} = \frac{2\Phi}{\mu S}\cdot \mu_r = \mu_r H_0$$

P7-13 (1)

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

Því er til vörpun } $\bar{A} \rightarrow \bar{A}' = \bar{A} + \bar{\nabla}\psi$ þ.a. $\bar{B}$ er óbreytt $\quad \bar{\nabla}\times\bar{A} = \bar{\nabla}\times\bar{A}'$

$\bar{E} = -\bar{\nabla}V - \partial_t \bar{A}$ Hvernig þarf vörpunin $V \rightarrow V'$ að vera til þ.a. $\bar{E}$ breytist ekki?

$$\bar{E}' = -\bar{\nabla}V' - \partial_t \bar{A}'$$

$$= -\bar{\nabla}V' - \partial_t\bar{A} - \bar{\nabla}\partial_t\psi = -\bar{\nabla}(V' + \partial_t\psi) - \partial_t\bar{A} = \bar{E}'$$

$\rightarrow \quad V' + \partial_t\psi = V$ þeas $\quad V' = V - \partial_t\psi$

Hvaða skilyrði þarf Φ að uppfylla svo $\bar{A}'$ og V' uppfylli áfram bylgjujöfnunum

$$\nabla^2 \bar{A}' - \mu\epsilon \partial_t^2 \bar{A} = -\mu \bar{J}$$

$$\nabla^2 V' - \mu\epsilon \partial_t^2 V' = -\frac{\rho}{\epsilon}$$

Lorentz skilyrðið tengdi A og V, reynum fyrir $\bar{A}'$ og V'

$$\bar{\nabla}\cdot\bar{A}' + \mu\epsilon\, \partial_t V' = 0$$

$$\bar{\nabla}\cdot(\bar{A} + \bar{\nabla}\Phi) + \mu\epsilon\,\partial_t(V - \partial_t\Phi) = 0$$

$$\longrightarrow \boxed{\nabla^2\Phi - \mu\epsilon\partial_t^2\Phi = 0} \leftarrow \text{bylgjujafna fyrir } \Phi$$

til þ.a. áfram gildi $\bar{\nabla}\cdot\bar{A} + \mu\epsilon\partial_t V = 0$

7-17

a) Relation between

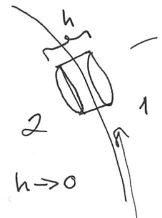
$$E_{1t} = E_{2t} \quad \text{and} \quad B_{1n} = B_{2n}$$

b) and between

$$\hat{a}_{n2}\cdot(\bar{D}_1 - \bar{D}_2) = \rho_s \quad \text{and} \quad \hat{a}_{n2}\times(\bar{H}_1 - \bar{H}_2) = \bar{J}_s$$

a) We can rewrite these two conditions as

$$\hat{a}_{n2}\cdot(\bar{B}_1 - \bar{B}_2) = 0 \quad \text{and} \quad \hat{a}_{n2}\times(\bar{E}_1 - \bar{E}_2) = 0$$



Consider a small volume. If

$$\hat{a}_{n2}\cdot(\bar{B}_1 - \bar{B}_2) = 0$$

at the surface then also

$$\hat{a}_{n2}\cdot(\partial_t\bar{B}_1\ \partial_t\bar{B}_2) = 0$$

Now we use $\bar{\nabla}\cdot(\hat{a}_{n2}\times\bar{E}) = \bar{E}\cdot(\bar{\nabla}\times\hat{a}_{n2}) - \hat{a}_{n2}\cdot(\bar{\nabla}\times\bar{E})$

to get

$$\hat{a}_{n2}\cdot(\partial_t\bar{B}_1 - \partial_t\bar{B}_2) \overset{\text{Maxwell – Faraday}}{=} -\hat{a}_{n2}\cdot(\bar{\nabla}\times\bar{E}_1 - \bar{\nabla}\times\bar{E}_2)$$

$$= \bar{\nabla}\cdot\left\{\hat{a}_{n2}\times(\bar{E}_1 - \bar{E}_2)\right\} = 0$$

integrate over the volume

use Gauss theorem to see that on the end surfaces

$$\hat{a}_{n2}\times(\bar{E}_1 - \bar{E}_1) = 0$$

$$\hat{a}_{n2}\cdot(\bar{D}_1 - \bar{D}_2) = \rho_s$$

$$\hat{a}_{n2}\cdot(\partial_t\bar{D}_1 - \partial_t\bar{D}_2) = \partial_t\rho_s$$

$$\hat{a}_{n2}\cdot\left(\bar{\nabla}\times\bar{H}_1 - \bar{J}_1 - (\bar{\nabla}\times\bar{H}_2 - \bar{J}_2)\right) = -\bar{\nabla}\cdot\bar{J}_s$$

$$\hat{a}_{n2}\cdot(\bar{\nabla}\times\bar{H}_1 - \bar{\nabla}\times\bar{H}_2) = -\bar{\nabla}\cdot\bar{J}_s + \hat{a}_{n2}\cdot(\bar{J}_1 - \bar{J}_2)$$

$$-\bar{\nabla}\cdot\left\{\hat{a}_{n2}\times(\bar{H}_1 - \bar{H}_2)\right\} = -\bar{\nabla}\cdot\bar{J}_s + \underbrace{\hat{a}_{n2}\cdot(\bar{J}_1 - \bar{J}_2)}_{=0\ \text{orthogonal}}$$

integral as before

$$\rightarrow \quad \hat{a}_{n2}\times(\bar{H}_1 - \bar{H}_2) = \bar{J}_s$$

P.7-30 (1)

Is the displacement current important in a good conductor?

a) Example Cu: $\epsilon_r = 1$, $\sigma = 5.70 \cdot 10^{7}$ S/m

Check at 100 GHz

Harmonic fields, Maxwell-Ampere

$$\overline{\nabla} \times \overline{H} = \overline{J} + i\omega\epsilon \overline{E} = (\sigma + i\omega\epsilon)\overline{E}$$

Ohmic current

$\overline{J} = \sigma \overline{E}$

Ohmic part

displacement part

$$\left| \frac{\text{Displacement}}{\text{Ohmic}} \right| = \frac{\omega\epsilon}{\sigma} \simeq 9.75 \cdot 10^{-8}$$

b) Source free good conductor, Eq. for $\overline{H}$ (2)

In a source free conductor, $\rho = 0$, but here $\overline{J} = \sigma\overline{E}$

The Maxwell eq. are

$$\overline{\nabla} \times \overline{E} = -i\omega\mu\overline{H}$$

$$\overline{\nabla} \times \overline{H} = i\omega\epsilon\overline{E} + \sigma\overline{E} = (i\omega\epsilon + \sigma)\overline{E} \approx \sigma\overline{E}$$

Due to a)

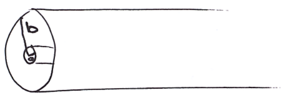
$$\overline{\nabla} \cdot \overline{E} = 0, \quad \overline{\nabla} \cdot \overline{H} = 0$$

$$\overline{\nabla} \times \overline{\nabla} \times \overline{H} = \sigma \, \overline{\nabla} \times \overline{E}$$

$$\overline{\nabla}(\overline{\nabla} \cdot \overline{H}) - \nabla^2 \overline{H} = -i\sigma\omega\mu\overline{H}$$

$$\rightarrow \nabla^2 \overline{H} - i\omega\mu\sigma\overline{H}$$

P8-19



V_0 - dc milli kjarna og kápu

I flæðir til viðnáms

Finna aflflæði með $\overline{S}$ í samása kapli

Gerum ráð fyrir línuhleðslu ρ_L á innri kjarna

Gauß lögmál gefur $\overline{E} = \hat{a}_r \frac{\rho_L}{2\pi\epsilon r}$

Þurfum að losna við ρ_L og fá V_0 í staðinn

$$V_0 = -\int_b^a \overline{E} \cdot d\overline{r} = \frac{\rho_L}{2\pi\epsilon} \ln\left(\frac{b}{a}\right)$$

$$\rightarrow \boxed{\overline{E} = \hat{a}_r \frac{V_0}{r\ln(b/a)}}$$

Lögmál Ampères gefur

$$\overline{H} = \hat{a}_\phi \frac{I}{2\pi r}$$

$$\overline{S} = \overline{E} \times \overline{H} = \hat{a}_z \frac{V_0 I}{2\pi r^2 \ln(b/a)}$$

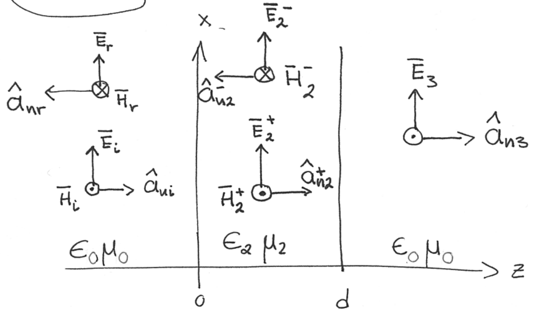
Nú þarf aflið sem flæðir um þverskurð kapals

$$P = \int_S \overline{S} \cdot d\overline{s} = \frac{V_0 I}{2\pi \ln(b/a)} \int_0^{2\pi} d\phi \int_a^b \left(\frac{1}{r^2}\right) r\,dr$$

$$= V_0 I$$

P.8-29

1



In book:

$$\overline{E}_1 = \hat{a}_x\left(E_{i0}e^{-i\beta_0 z} + E_{r0}e^{i\beta_0 z}\right)$$
$$\overline{E}_2 = \hat{a}_x\left(E_2^+ e^{-i\beta_2 z} + E_2^- e^{i\beta_2 z}\right)$$
$$\overline{E}_3 = \hat{a}_x E_{t0}\, e^{-i\beta_0 z}$$
$$\overline{H}_1 = \frac{\hat{a}_y}{\eta_0}\left(E_{i0}e^{-i\beta_0 z} - E_{r0}e^{i\beta_0 z}\right)$$
$$\overline{H}_2 = \frac{\hat{a}_y}{\eta_2}\left(E_2^+ e^{-i\beta_2 z} - E_2^- e^{i\beta_2 z}\right)$$
$$\overline{H}_3 = \frac{\hat{a}_y}{\eta_0} E_{t0}\, e^{-i\beta_0 z}$$

a) find E_{r0}, E_2^+, E_2^-, E_{t0} in terms of $E_{i0}, d, \epsilon_2, \mu_2$

Now the boundary conditions are, (no surface charge or currents) (only tangential fields) (lossless dielectric slab):

$$E_{1t}(0) = E_{2t}(0) \;\rightarrow\; \overline{E}_1(0) = \overline{E}_2(0) \qquad (1)$$
$$E_{2t}(d) = E_{3t}(d) \;\rightarrow\; \overline{E}_2(d) = \overline{E}_3(d) \qquad (2)$$

2

we also have generally $\hat{a}_{n2}\times(\overline{H}_1 - \overline{H}_2) = 0$ here translating into:

$$(3) \quad \overline{H}_1(0) = \overline{H}_2(0) \quad \text{and} \quad \overline{H}_2(d) = \overline{H}_3(d) \quad (4)$$

1: $E_{i0} + E_{r0} = E_2^+ + E_2^-$

2: $E_2^+ e^{-i\beta_2 d} + E_2^- e^{i\beta_2 d} = E_{t0}\, e^{-i\beta_0 d}$

3: $\frac{1}{\eta_0}\left(E_{i0} - E_{r0}\right) = \frac{1}{\eta_2}\left(E_2^+ - E_2^-\right)$

4: $\frac{1}{\eta_2}\left(E_2^+ e^{-i\beta_2 d} - E_2^- e^{i\beta_2 d}\right) = \frac{1}{\eta_0}E_{t0}\, e^{-i\beta_0 d}$

We know that

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}, \quad \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}, \quad \beta_0 = \frac{\omega}{c}, \quad \beta_2 = \omega\sqrt{\mu_2\epsilon_2}$$

3

Transform

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & e^{-i\beta_2 d} & e^{i\beta_2 d} & -e^{-i\beta_0 d} \\ -\frac{1}{\eta_0} & -\frac{1}{\eta_2} & \frac{1}{\eta_2} & 0 \\ 0 & \frac{e^{-i\beta_2 d}}{\eta_2} & -\frac{e^{i\beta_2 d}}{\eta_2} & -\frac{e^{-i\beta_0 d}}{\eta_0} \end{pmatrix} \begin{pmatrix} E_{r0} \\ E_2^+ \\ E_2^- \\ E_{t0} \end{pmatrix} = \begin{pmatrix} -E_{i0} \\ 0 \\ -\frac{E_{i0}}{\eta_0} \\ 0 \end{pmatrix}$$

I use wxMaxima to get

$$E_{r0} = \frac{\left\{(\eta_2^2 - \eta_0^2)e^{2i\beta_2 d} - (\eta_2^2 - \eta_0^2)\right\}E_{i0}}{(\eta_2+\eta_0)^2 e^{2i\beta_2 d} - (\eta_2 - \eta_0)^2}$$

4

$$E_2^+ = \frac{2(\eta_2^2 + \eta_0\eta_2)e^{2i\beta_2 d}\, E_{i0}}{(\eta_2+\eta_0)^2 e^{2i\beta_2 d} - (\eta_2-\eta_0)^2}$$

$$E_2^- = \frac{2(\eta_2^2 - \eta_0\eta_2)\,E_{i0}}{(\eta_2+\eta_0)^2 e^{2i\beta_2 d} - (\eta_2-\eta_0)^2}$$

$$E_{t0} = \frac{4\eta_0\eta_2\, e^{i\beta_2 d}\, E_{i0}}{\left\{(\eta_2+\eta_0)^2 e^{2i\beta_2 d} - (\eta_2-\eta_0)^2\right\}e^{-i\beta_0 d}}$$

If $\eta_2 = \eta_0 \;\rightarrow\; \begin{cases} E_{r0} = 0, \quad E_2^- = 0, \quad E_2^+ = E_{i0} \\ \text{and} \quad E_{t0} = E_{i0} \end{cases}$ as has to be

5

b) Is there a reflection at $z = 0$ if $d = \frac{\lambda_2}{4}$? If $d = \frac{\lambda_2}{2}$?

If $d = \frac{\lambda_2}{4}$ $\rightarrow$ $\beta_2 = \frac{2\pi}{\lambda_2} = \frac{2\pi}{4d} = \frac{\pi}{2d}$

$\rightarrow$ $\beta_2 d = \frac{\pi}{2}$

$$E_{ro} = \frac{\left\{(\eta_2^2-\eta_0^2)(-1) - (\eta_2^2-\eta_0^2)\right\} E_{io}}{(\eta_2+\eta_0)^2(-1) - (\eta_2-\eta_0)^2} = \frac{(\eta_2^2-\eta_0^2)E_{io}}{(\eta_2^2+\eta_0^2)}$$

If $d = \frac{\lambda_2}{2}$ $\rightarrow$ $\beta_2 = \frac{2\pi}{\lambda_2} = \frac{2\pi}{2d} = \frac{\pi}{d}$

$\rightarrow$ $\beta_2 d = \pi$

6

$$E_{ro} = \frac{\left\{(\eta_2^2-\eta_0^2) - (\eta_2^2-\eta_0^2)\right\} E_{io}}{(\eta_2+\eta_0)^2 - (\eta_2-\eta_0)^2} = 0$$

The reflection from $z=d$ interferes destructively with the reflection from $z=0$

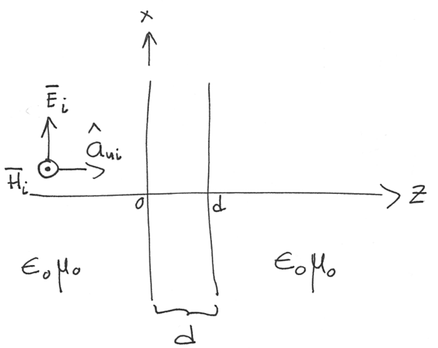
$\rightarrow$ no reflection from $z=0$ in this case

7

8-33

$$\bar{E}_i(z) = \hat{a}_x E_{io} e^{-i\beta_0 z}$$

normal on a thin Cu-plate with thickness d



Good conductor

$$\alpha_2 = \beta_2 = \sqrt{\pi f \mu_2 \sigma_2} = \frac{1}{\delta_2}$$

$$\eta_2 = (1+i)\frac{\alpha_2}{\sigma_2}$$

$$k_2 = \beta_2 - i\alpha_2 = (1-i)\frac{1}{\delta_2}$$

$\eta_2 \ll \eta_0$ since $\frac{\alpha_2}{\sigma_2} = \sqrt{\frac{\pi f \mu_2}{\sigma_2}}$

$$= \sqrt{\frac{\pi \cdot 10 \cdot 10^6 \; 4\pi \cdot 10^{-7}}{5.8 \cdot 10^7}} \sim 8.25 \cdot 10^{-4}$$

for $f = 10 \cdot 10^6$ Hz

8

Neglect multiple reflections find:

a) E_2^+, H_2^+ using 8-29

$$\eta_2 H_2^+ = E_2^+ = \frac{2(\eta_2^2 - \eta_0\eta_2)e^{2\alpha_2 d}e^{2i\beta_2 d} E_{io}}{(\eta_2+\eta_0)^2 e^{2\alpha_2 d}e^{2i\beta_2 d} - (\eta_2-\eta_0)^2}$$

$$\simeq - \frac{2\eta_0\eta_2 e^{2\alpha_2 d} e^{2i\beta_2 d} E_{io}}{\eta_0^2\left(e^{2\alpha_2 d}e^{2i\beta_2 d} - 1\right)}$$

$$= -i\left(\frac{\eta_2}{\eta_0}\right)\frac{e^{\alpha_2 d}e^{i\beta_2 d} E_{io}}{\sin((\beta_2 - i\alpha_2)d)}$$

(9)

b) E_2^-, H_2^-

$$-\eta_2 H_2^- = E_2^- \simeq -i\left(\frac{\eta_2}{\eta_0}\right)\frac{e^{-\alpha_2 d}e^{-i\beta_2 d}\,E_{io}}{\sin((\beta_2 - i\alpha_2)d)}$$

c) E_{30}, H_{30}

$$\eta_0 H_{30} = E_{30} = E_{to} \simeq -i\left(\frac{\eta_2}{\eta_0}\right)\frac{2e^{i\beta_0 d}\,E_{io}}{\sin((\beta_2 - i\alpha_2)d)}$$

d) $(\mathcal{S}_{ave})_3 / (\mathcal{S}_{ave})_i$

(10)

$$E_{ro} = \frac{(\eta_2^2-\eta_0^2)\left(e^{2\alpha_2 d}e^{2i\beta_2 d}-1\right)E_{io}}{(\eta_2+\eta_0)^2 e^{2\alpha_2 d}e^{2i\beta_2 d} - (\eta_2-\eta_0)^2}$$

$$= \frac{(\eta_2^2-\eta_0^2)\left\{e^{\alpha_2 d}e^{i\beta_2 d} - e^{-\alpha_2 d}e^{-i\beta_2 d}\right\}E_{io}}{(\eta_2+\eta_0)^2 e^{\alpha_2 d}e^{i\beta_2 d} - (\eta_2-\eta_0)^2 e^{-\alpha_2 d}e^{-i\beta_2 d}}$$

$$= \frac{(\eta_2^2-\eta_0^2)\sin((\beta_2-i\alpha_2)d)\,E_{io}\,2i}{(\eta_2^2+\eta_0^2)\,2i\sin((\beta_2-i\alpha_2)d) + 4\eta_2\eta_0\cos((\beta_2-i\alpha_2)d)}$$

$$\simeq -\frac{E_{io}}{1-2i\left(\frac{\eta_2}{\eta_0}\right)\cot((\beta_2-i\alpha_2)d)}$$

(11)

$$(\bar{\mathcal{S}}_{ave})_1 = \frac{1}{2}\mathcal{Re}\left\{(\bar{E}_{io}\times\bar{H}_{io}^*) - (\bar{E}_{ro}\times\bar{H}_{ro}^*)\right\}\hat{a}_z$$

$$E_{ro} \simeq \frac{-E_{io}}{1-2i\left(\frac{\eta_2}{\eta_0}\right)\cot((\beta_2-i\alpha_2)d)}$$ ← use (GR 1.313.11)

$$= -\frac{E_{io}}{1-2i\left(\frac{\eta_2}{\eta_0}\right)\frac{1+i\tan(\beta_2 d)\tanh(\alpha_2 d)}{\tan(\beta_2 d)-i\tanh(\alpha_2 d)}}$$

$$\simeq -E_{io}\left\{1+2i\left(\frac{\eta_2}{\eta_0}\right)\frac{1+i\tan(\beta_2 d)\tanh(\alpha_2 d)}{\tan(\beta_2 d)-i\tanh(\alpha_2 d)}\right\}$$

(12)

If now $E_{ro} \simeq -E_{io} - E_{io}i\Delta$

then
$$(\mathcal{S}_{ave})_1 = \frac{1}{2}\mathcal{Re}\left\{|E_{io}|^2\frac{1}{\eta_0} - |E_{io}|^2\frac{1}{\eta_0} + |E_{io}|^2\frac{1}{\eta_0}(\Delta+\Delta^*)\right\}$$

With
$$\Delta = 2\left(\frac{\eta_2}{\eta_0}\right)\frac{1+i\tan(\beta_2 d)\tanh(\alpha_2 d)}{\tan(\beta_2 d)-i\tanh(\alpha_2 d)}$$

use (GR 1.313.3)

$$(\mathcal{S}_{ave})_3 = \frac{1}{2\eta_0}|E_{30}|^2 = \frac{1}{2\eta_0}\frac{|\eta_2|^2}{\eta_0^2}\frac{4|E_{io}|^2}{\left\{\sin(\beta_2 d)\cosh(\alpha_2 d)+\cos(\beta_2 d)\sinh(\alpha_2 d)\right\}^2}$$

$$(\mathcal{P}_{ave})_i = \frac{1}{2\eta_0}|E_{id}|^2$$

$$\rightarrow \frac{(\mathcal{P}_{ave})_3}{(\mathcal{P}_{ave})_i} = \frac{|\eta_2|^2}{\eta_0^2} \frac{4}{\left\{\operatorname{Sin}(\beta_2 d)\operatorname{Cosh}(\alpha_2 d) + \operatorname{Cos}(\beta_2 d)\operatorname{Sinh}(\alpha_2 d)\right\}^2}$$

$$\frac{|\eta_2|^2}{\eta_0^2} = \frac{|(1+i)|^2 \alpha_2^2}{\eta_0^2 \, \sigma_2^2} = \frac{2\alpha_2^2}{\eta_0^2 \sigma_2^2}$$

$$\rightarrow \frac{(\mathcal{P}_{ave})_3}{(\mathcal{P}_{ave})_i} \simeq \frac{8}{\eta_0^2}\left(\frac{\alpha_2}{\sigma_2}\right)^2 \frac{1}{\left\{\operatorname{Sin}(\beta_2 d)\operatorname{Cosh}(\alpha_2 d) + \operatorname{Cos}(\beta_2 d)\operatorname{Sinh}(\alpha_2 d)\right\}^2}$$

$$\simeq 1.8 \cdot 10^{-11} \quad \text{for} \quad f = 10^7\ \mathrm{Hz},\ d = \delta$$