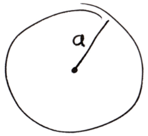
①  Gegnheil rafsvarandi kúla ~~með~~ skautun $\bar{P}(R) = P_0 \left(\frac{R}{a}\right)^2 \hat{a}_R$

① Finna jafngilda bol- og yfirborðsþéttleika

bolur:

$$\rho_P = -\bar{\nabla}\cdot\bar{P} \quad \text{aðeins fall af } R$$

$$\rightarrow \rho_P = -\frac{1}{R^2}\partial_R\left\{R^2 P_0 \left(\frac{R}{a}\right)^2\right\}$$ kúluhnit ef $R<a$

$$= -\frac{1}{R^2}\partial_R\left\{R^4\right\}\frac{P_0}{a^2}$$

$$= -4\left(\frac{R}{a^2}\right)P_0 \quad \text{ef } R<a$$

og 0 fyrir $R>a$

② Yfirborð:

$$\rho_{ps} = \bar{P}\cdot\hat{a}_n\Big|_{R=a} = \bar{P}\cdot\hat{a}_R\Big|_{R=a} = P_0$$

③ Heildar jafngilda hleðslan

$$Q_{ps} = \oint ds\,\rho_{ps} = 4\pi a^2 P_0 \quad \text{fyrir yfirborðið}$$

$$Q_P = \int_V dv\,\rho_P = -\frac{4P_0}{a^2}4\pi\int R^2 dR\,R = -\frac{4P_0}{a^2}4\pi\frac{a^4}{4}$$

$$= -4\pi a^2 P_0$$

$\rightarrow Q_{ps} + Q_P = 0$, er það tilviljun?

Lítum á

$$Q_P = \int_V dv\,\rho_P = -\int_V dv\,\bar{\nabla}\cdot\bar{P}$$

notum setningu Gauß $\longrightarrow -\oint_S d\bar{s}\cdot\bar{P} = -Q_{ps}$

$\rightarrow Q_P + Q_{ps} = 0$ alltaf

④ Rafsviðið metið?

Jafngilda hleðsludreifingarnar eru kúlusamhverfar, óháðar hornunum $\rightarrow$ notum lögmál Gauß og finnum fyrst sviðið

því fáumst við ekki við D hér hjálparsviðið.

Innan kúlu, $R<a$ $\quad \oint_{S(R)} \bar{E}\cdot d\bar{s} = \frac{Q(R)}{\epsilon_0}$

Hleðslan innan yfirborðs $S(R)$

$$Q(R) = \int_{R'<R} dv'\,\rho(R') = 4\pi\int_0^R R'^2 dR'\left\{-4\frac{R'P_0}{a^2}\right\}$$

$$= -4\pi\frac{4P_0}{a^2}\int_0^R dR'\,(R')^3 = -4\pi\frac{4P_0}{a^2}\frac{R^4}{4} = -4\pi P_0\frac{R^4}{a^2}$$

Notum í lögmál Gauß

$$4\pi R^2 E(R) = -\frac{4\pi R^4}{\epsilon_0 a^2}P_0 \rightarrow \boxed{\bar{E}(R) = -\left(\frac{R}{a}\right)^2\frac{P_0}{\epsilon_0}\hat{a}_R \quad R<a}$$

utan kúlu $Q=0$, kúlusamhverfa

$$\rightarrow \boxed{\bar{E} = 0, \quad R>a}$$

5

$$\bar{E} = -\bar{\nabla} V = -\partial_R V(R)$$ vegna kúlusamhverfu

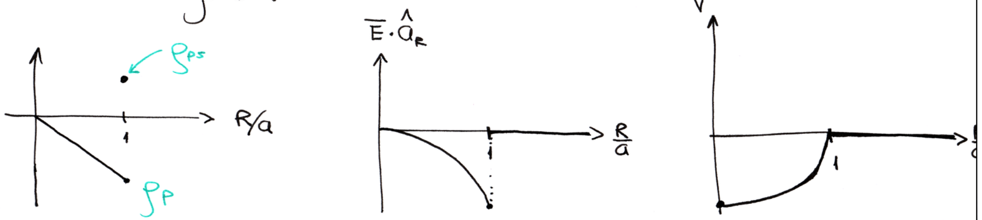
setjum $V=0$ þ. $R \to \infty$ $\to$ $V(R)=0$ fyrir $R>a$

og $V(R) = \frac{a}{3}\left(\frac{R}{a}\right)^3 \frac{P_0}{\epsilon_0} + C$

$V(R)\Big|_{R=a} = 0$ því $V(R)$ er samfellt í yfirborði kúlunnar

$$\to V(R) = \frac{P_0 a}{3\epsilon_0}\left\{\left(\frac{R}{a}\right)^3 - 1\right\}, \quad R<a$$

⑥ Rissum myndir



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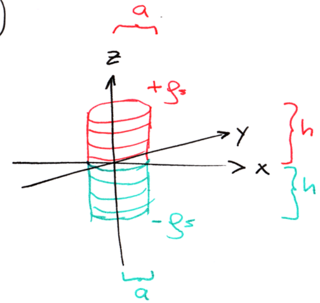
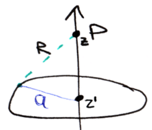
②



① Finna V á z-ás, alls staðar

Athugum hring:



Fjarlægðin frá hringnum að P er alls staðar sú sama

$$dV = \frac{\rho_l}{4\pi\epsilon_0 R}$$

$$\to V = \oint \frac{\rho_l dl'}{4\pi\epsilon_0 R} = \frac{\rho_l}{4\pi\epsilon_0 R}\oint dl'$$

$$= \frac{\rho_l a}{2\epsilon_0 R} = \frac{\rho_l a}{2\epsilon_0 \sqrt{a^2+(z-z')^2}}$$

því má finna V fyrir sívalning m.þ.a. „summa" upp fyrir hringi

$$dV(z) = \frac{\rho_s a dz'}{2\epsilon_0\sqrt{a^2+(z-z')^2}}$$

7

Efri sívalningur $(z>h)$

$$V_+(z) = \int_0^h \frac{\rho_s a dz'}{2\epsilon_0\sqrt{a^2+(z-z')^2}} = \frac{\rho_s a}{2\epsilon_0}\int_0^h \frac{dz'}{\sqrt{a^2+(z-z')^2}}$$

(GR 2.271.4)

Gerum heildið með öðrum og athugum seinna hvernig það lítur út fyrir mismunandi svæði

$(z>h)$ ofan

$(z<0)$ neðan

$(h>z>0)$ inni

$$= \frac{\rho_s a}{2\epsilon_0}\left\{-\ln\left[(z-z') + \sqrt{a^2+(z-z')^2}\right]\Bigg|_0^h\right\}$$

$$= \frac{\rho_s a}{2\epsilon_0}\left\{-\ln\left[(z-h)+\sqrt{a^2+(z-h)^2}\right] + \ln\left[z+\sqrt{a^2+z^2}\right]\right\}$$

$$= \frac{\rho_s a}{2\epsilon_0}\left[-\ln\left\{\frac{(z-h)+\sqrt{a^2+(z-h)^2}}{z+\sqrt{a^2+z^2}}\right\}\right]$$ ef $z>h$

hér væri hreinlegast að nota innsetn.
$z-z' = x$, $dx = -dz'$

8

$V_+(z)$ þegar $z<0$

breytuskipti
$z'-z = x$
$dx = dz'$

$$V_+(z) = \int_0^h \frac{\rho_s a dz'}{2\epsilon_0\sqrt{a^2+(z'-z)^2}} = \int_{-z}^{h-z}\frac{\rho_s a dx}{2\epsilon_0\sqrt{a^2+x^2}}$$

$$= \frac{\rho_s a}{2\epsilon_0}\left\{\ln\left[x+\sqrt{a^2+x^2}\right]\Bigg|_{-z}^{h-z}\right\}$$

$$= \frac{\rho_s a}{2\epsilon_0}\left\{\ln\left[\frac{(h-z)+\sqrt{a^2+(h-z)^2}}{\sqrt{a^2+z^2}-z}\right]\right\}$$

þegar $h>z>0$

$$V_+(z) = \int_0^z \frac{\rho_s a dz'}{2\epsilon_0\sqrt{a^2+(z-z')^2}} + \int_z^h \frac{\rho_s a dz'}{2\epsilon_0\sqrt{a^2+(z'-z)^2}}$$

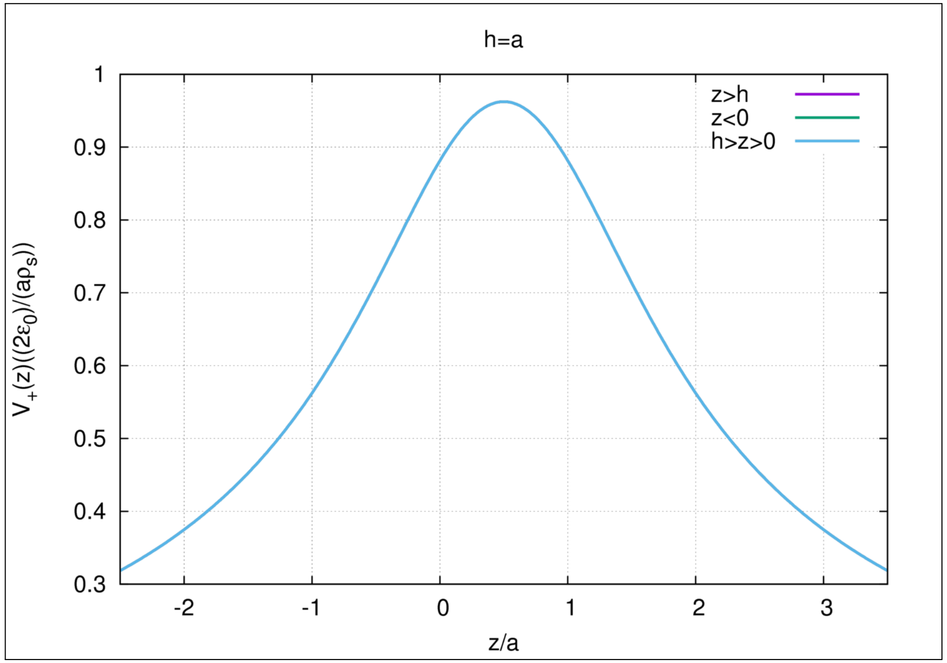
$$= \frac{\rho_s a}{2\epsilon_0}\left\{ -\ln\left[x+\sqrt{a^2+x^2}\right]\Big|_z^0 + \ln\left[x+\sqrt{a^2+x^2}\right]\Big|_0^{h-z}\right\}$$

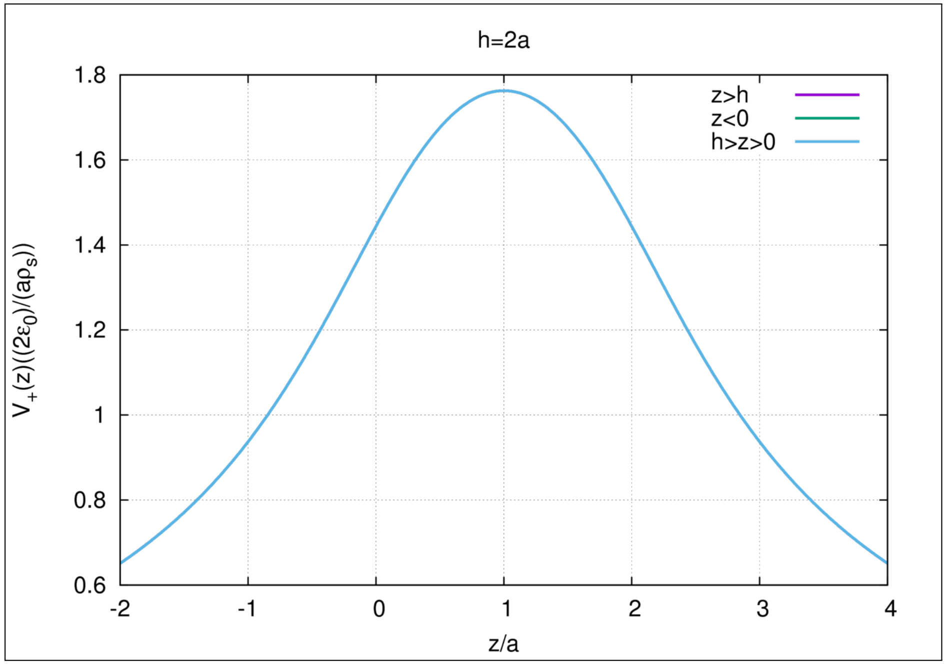
$$= \frac{\rho_s a}{2\epsilon_0}\left\{-\ln a + \ln\left[z+\sqrt{a^2+z^2}\right] + \ln\left[(h-z)+\sqrt{a^2+(h-z)^2}\right] - \ln a\right\}$$

$$\rightarrow V_+(z) = \frac{\rho_s a}{2\epsilon_0}\left\{\ln\left[\frac{\left(z+\sqrt{a^2+z^2}\right)\left((h-z)+\sqrt{a^2+(h-z)^2}\right)}{a^2}\right]\right\}$$

þegar $h>z>0$

Á næstu tveimur síðum sést að jöfnurnar fyrir $V_+(z)$ eru í raun jafngildar á öllum svæðunum





(2) Því kemur ekki á óvart að

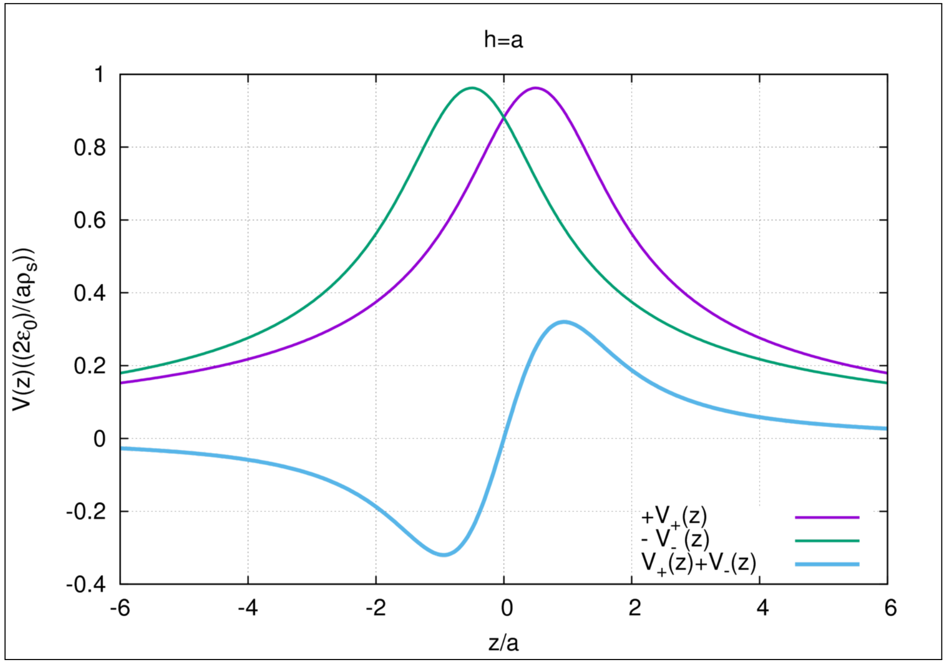
$$\bar{E} = -\hat{a}_z \frac{\partial V_+(z)}{\partial z} = \hat{a}_z \frac{\rho_s a}{2\epsilon_0}\left\{\frac{1}{\sqrt{a^2+(z-h)^2}} - \frac{1}{\sqrt{a^2+z^2}}\right\}$$

á öllum svæðunum

(3) $$V_-(z) = \frac{\rho_s a}{2\epsilon_0}\left[-\ln\left\{\frac{(z+h)+\sqrt{a^2+(z+h)^2}}{z+\sqrt{a^2+z^2}}\right\}\right]$$

hér er hleðslan $-\rho_s$

Á næstu síðu er graf af $V(z) = V_+(z) + V_-(z)$



(4)

$$V(z) = -\frac{\rho_s a}{2\epsilon_0}\left[\ln\left\{\frac{\left((z-h)+\sqrt{a^2+(z-h)^2}\right)\left((z+h)+\sqrt{a^2+(z+h)^2}\right)}{\left(z+\sqrt{a^2+z^2}\right)^2}\right\}\right]$$

Viljum finna ð fellu formið þegar $|z| >> h, a$

$$V(z) \to -\frac{\rho_s a}{2\epsilon_0}\left[\ln\left\{\frac{\left((1-\frac{h}{z})+\sqrt{(\frac{a}{z})^2+(1-\frac{h}{z})^2}\right)\left((1+\frac{h}{z})+\sqrt{(\frac{a}{z})^2+(1+\frac{h}{z})^2}\right)}{\left(1+\sqrt{(\frac{a}{z})^2+1}\right)^2}\right.\right.$$

$$\left(1-\frac{h}{z}\right)+\sqrt{\left(\frac{a}{z}\right)^2+\left(1-\frac{h}{z}\right)^2} \to 1-\frac{h}{z}+1+\frac{1}{2}\left(\frac{a}{z}\right)^2+\frac{1}{2}\left(\frac{h}{z}\right)^2-\frac{h}{z}$$

notum WxMaxima til ð finna röð fyrir $\frac{a}{z}, \frac{h}{z}$, þá fæst

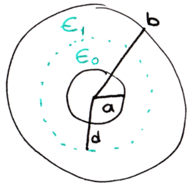
$$V(z) \to \frac{\rho_s a}{2\epsilon_0}\left\{\frac{h^2}{z^2}-\frac{(3h^2a^2-h^4)}{2z^4}+\dots\right\}$$

Tvískauts ð fella

1

① Kúluþéttir

$|V_{ab}| = V_0$



Kúlusamhverfa

① Finnum rýmd.

Heppilegast er að nota $\oint_S \bar{D}\cdot d\bar{s} = Q$

þar sem við gefum okkur að hleðslan á innri skelinni (frjálsa) sé Q

Því fæst fyrir $a < R < b$ að

$$4\pi R^2 \bar{D}\cdot \hat{a}_R = Q \quad \rightarrow \quad \bar{D}(R) = \frac{\hat{a}_R Q}{4\pi R^2}$$

Rýmd er skilgreind þ.a. $Q = CV$, við þurfum því að finna tengsl Q og V hér.

$$\bar{E} = -\bar{\nabla} V \quad \rightarrow \quad V_R - V_a = -\int_a^R \bar{E}\cdot d\bar{l}$$

2

$$\bar{D}(R) = \hat{a}_R \frac{Q}{4\pi R^2} \qquad a < R < b$$

$$\bar{E}(R) = \begin{cases} \hat{a}_R \dfrac{Q}{4\pi\epsilon_0 R^2} & a<R<d \\[2ex] \hat{a}_R \dfrac{Q}{4\pi\epsilon_1 R^2} & d<R<b \end{cases}$$

Veljum heildunarveg samsíða $\hat{a}_R$

$$\rightarrow \quad V_b - V_a = -\int_a^d \frac{Q\,dR}{4\pi R^2 \epsilon_0} - \int_d^b \frac{Q\,dR}{4\pi R^2 \epsilon_1} = -\frac{Q}{4\pi}\left[\int_a^d \frac{dR}{\epsilon_0 R^2} + \int_d^b \frac{dR}{\epsilon_1 R^2}\right]$$

$$= -\frac{Q}{4\pi}\left\{\left(\frac{1}{a} - \frac{1}{d}\right)\frac{1}{\epsilon_0} + \left(\frac{1}{d} - \frac{1}{b}\right)\frac{1}{\epsilon_1}\right\}$$

3

$$V_b - V_a = -\frac{Q}{4\pi\epsilon_0}\left[\frac{d-a}{ad} + \frac{\epsilon_0}{\epsilon_1}\,\frac{b-d}{db}\right]$$

$$C = \frac{Q}{|V_b - V_a|} = \frac{4\pi\epsilon_0}{\dfrac{d-a}{ad} + \dfrac{\epsilon_0}{\epsilon_1}\,\dfrac{b-d}{db}}$$

Hér sést þetta jafna fyrir kúluþétti án rafsvara þegar $d \rightarrow b$

---

② Frjálsar hleðslur eru $+Q$ á innri skelinni og $-Q$ á ytri st.

getinn var spennumunurinn $V_0 = |V_b - V_a|$

$$Q = C|V_b - V_a| = CV_0 = \frac{4\pi\epsilon_0 V_0}{\dfrac{d-a}{ad} + \dfrac{\epsilon_0}{\epsilon_1}\,\dfrac{b-d}{db}}$$

4

Jafngildar skautaðar hleðslur

Skautunarsviðið er skilgreint sem

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P} \quad \rightarrow \quad \bar{P} = \bar{D} - \epsilon_0 \bar{E}$$

Skautunar hleðslurnar svo sem

$$\rho_{ps} = \bar{P}\cdot\hat{a}_n$$

$$\rho_p = -\bar{\nabla}\cdot\bar{P}$$

Í kúluhnitum er

$$\bar{\nabla}\cdot\bar{P}(R) = \frac{1}{R^2}\partial_R\left[R^2 P(R)\right]$$

$\rightarrow \quad \bar{\nabla}\cdot\bar{P} = 0$ í okkar tilfelli fyrir $b > R > d$ og $d > R > a$

Skoðum því yfirborðs skautunina

miðað við rafsvara

$$\rho_{ps}(a^+) = -\hat{a}_R\cdot\bar{P}(a^+)$$

(5)

$$\bar P(a^+) = \bar D(a^+) - \epsilon_0 \bar E(a^+) = \epsilon_0\left\{\bar E(a^+) - \bar E(a^+)\right\} = 0$$

→ $\rho_{ps}(a^+) = 0$, enda er $\epsilon = \epsilon_0$ við þá skel

$$\rho_{ps}(b^-) = +\hat a_R \cdot \bar P(b^-) = +\left\{\frac{Q}{4\pi b^2} - \frac{Q}{4\pi b^2}\frac{\epsilon_0}{\epsilon_1}\right\}$$

miðað við rafsvara

$$= +\frac{Q}{4\pi b^2}\left(1 - \frac{\epsilon_0}{\epsilon_1}\right) > 0$$

Þurfum því að athuga hinn enda rafsvarans við $R = d$

$$\rho_{ps}(d^+) = -\hat a_R \cdot \bar P(d^+) = -\left\{\frac{Q}{4\pi d^2} - \frac{Q}{4\pi d^2}\frac{\epsilon_0}{\epsilon_1}\right\}$$

$$= -\frac{Q}{4\pi d^2}\left(1 - \frac{\epsilon_0}{\epsilon_1}\right) < 0$$

þéttleiki skautaðra hleðslna á rafsvaranum er mismunandi en heildar skautað hleðslan sitthvoru megin á rafsvaranum er jafn mikil með sitthvort formerkið.

(6)

(3) Rafstöðukrafturinn á ytri skelina?

spennunni V_0 er haldið fastri → $\bar F_v = \bar\nabla W_e$

$$W_e = \frac{1}{2}\int dv\, \bar D \cdot \bar E = \frac{1}{2} C V_0^2$$

$$= \frac{2\pi\epsilon_0 V_0^2}{\frac{d-a}{ad} + \frac{\epsilon_0}{\epsilon_1}\frac{b-d}{db}}$$

Gerum ráð fyrir að hægt sé að hnika til aðeins geisla ytri skeljarinnar

$$\to \left(F_v\right)_R = \frac{1}{2} V_0^2 \partial_b C(b)$$

(7)

$$\to \left(F_v\right)_R = -\frac{V_0^2}{2}\,\frac{\frac{\epsilon_0}{db\,\epsilon_1}\left\{1 - \frac{b-d}{b}\right\}}{\left\{\frac{d-a}{ad} + \frac{\epsilon_0}{\epsilon_1}\frac{b-d}{db}\right\}^2} < 0$$

→ Krafturinn er inn á við, innri skelin togar í þá ytri til sín.

(2) Er til hleðsludreifing á kúluyfirborði þ.a. á miðbaug kúlunnar væri $\bar E \sim \hat a_\phi$

Í rafstöðufræðinni eru tvær jöfnur

$$\bar\nabla \cdot \bar D = \rho, \qquad \bar\nabla \times \bar E = 0$$

(8)

Á heildisformi er seinni jafnan

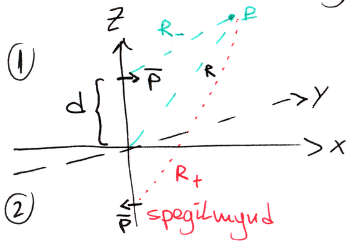
$$\oint_C \bar E \cdot d\bar l = 0$$

Ef $\bar E \sim \hat a_\phi \to \oint_C \bar E \cdot d\bar l \neq 0$

→ ekki er til þannig rafstöðuhleðsla!

①

① Raftvískaut yfir leiðandi sléttu



Setjum hnita kerfi eins og myndin sýnir

$$R_- = |\bar R - \bar d| , \quad R_+ = |\bar R + \bar d|$$

$$\bar p = p_0 \cdot \hat a_x$$

$$V = V_+ + V_- = -\frac{\hat a_x \cdot \hat a_R p_0}{4\pi\epsilon_0 R_+^2} + \frac{\hat a_x \cdot \hat a_R p_0}{4\pi\epsilon_0 R_-^2}$$

$$= \frac{p_0 \hat a_x \cdot \hat a_R}{4\pi\epsilon_0}\left[-\frac{1}{R_+^2} + \frac{1}{R_-^2}\right]$$

notum kúluhnit

$$R_\pm^2 = R^2 + d^2 \pm 2Rd\cos\theta$$

$$\hat a_x \cdot \hat a_R = \sin\theta\cos\phi$$

②

$$V(R,\phi,\theta) = \frac{p_0 \sin\theta\cos\phi}{4\pi\epsilon_0}\left\{\frac{1}{R^2+d^2-2Rd\cos\theta} - \frac{1}{R^2+d^2+2Rd\cos\theta}\right\}$$

$$= \frac{p_0 4Rd\cos\theta\sin\theta\cos\phi}{4\pi\epsilon_0\left\{(R^2+d^2)^2 - (2Rd\cos\theta)^2\right\}}$$

$$= \frac{2p_0 Rd\sin(2\theta)\cos\phi}{4\pi\epsilon_0\left\{(R^2+d^2)^2 - (2Rd\cos\theta)^2\right\}}$$

---

③ skoðum strax að fella formið $R >> d$

$$\rightarrow V(R,\phi,\theta) \rightarrow \frac{p_0 d\sin(2\theta)\cos\phi}{2\pi\epsilon_0 R^3}$$

③

V fellur hraðar, en fyrir eitt tvískaut, $\frac{1}{R^3}$ → fjórskaut

Horndreifingin í θ-átt er $\sim \sin(2\theta)$

→ fjórskaut sett saman úr tveimur tvískautum, tvískauti og spegilmynd þess.

Rafsvið $\quad \bar E = -\bar\nabla V = -\hat a_R \frac{\partial V}{\partial R} - \hat a_\theta \frac{\partial V}{R\partial\theta} - \hat a_\phi \frac{1}{R\sin\theta}\frac{\partial V}{\partial\phi}$

Notum $\hat a_{n2}\cdot(\bar D_1 - \bar D_2) = \rho_s \quad \rightarrow \quad \hat a_z \cdot \epsilon_0 \bar E = \rho_s$

Jafnan fyrir V er í kúluhnitum, en kartískt væri heppilegri, en jafnan er þá voldugt flókin. En munum að

$$E_z = E_R\cos\theta - E_\theta\sin\theta$$

④

og í x-y-sléttunni er $\theta = \frac{\pi}{2}$ og því þurfum við aðeins E_θ, enda er $\hat a_\theta = -\hat a_z$ í sléttunni

$$E_z = -E_\theta = \bar\nabla_\theta V = \frac{1}{R}\frac{\partial V}{\partial\theta}$$

$$= \frac{p_0 d\cos(2\theta)\cos\phi}{\pi\epsilon_0\left\{(R^2+d^2)^2 - (2Rd\cos\theta)^2\right\}} - \frac{2R^3d^2p_0\sin(2\theta)\cos\phi\cos\theta\sin\theta}{\pi\epsilon_0\left\{(R^2+d^2)^2 - (2Rd\cos\theta)^2\right\}^2}$$

$$\rightarrow E_z(R,\tfrac{\pi}{2},\phi) = -\frac{p_0 d\cos\phi}{\pi\epsilon_0\left\{(R^2+d^2)^2\right\}} = -\frac{p_0 d\cos\phi}{\pi\epsilon_0(R^2+d^2)^2}$$

$$\rightarrow \rho_s(R,\phi) = -\frac{p_0 d\cos\phi}{\pi(R^2+d^2)^2}$$

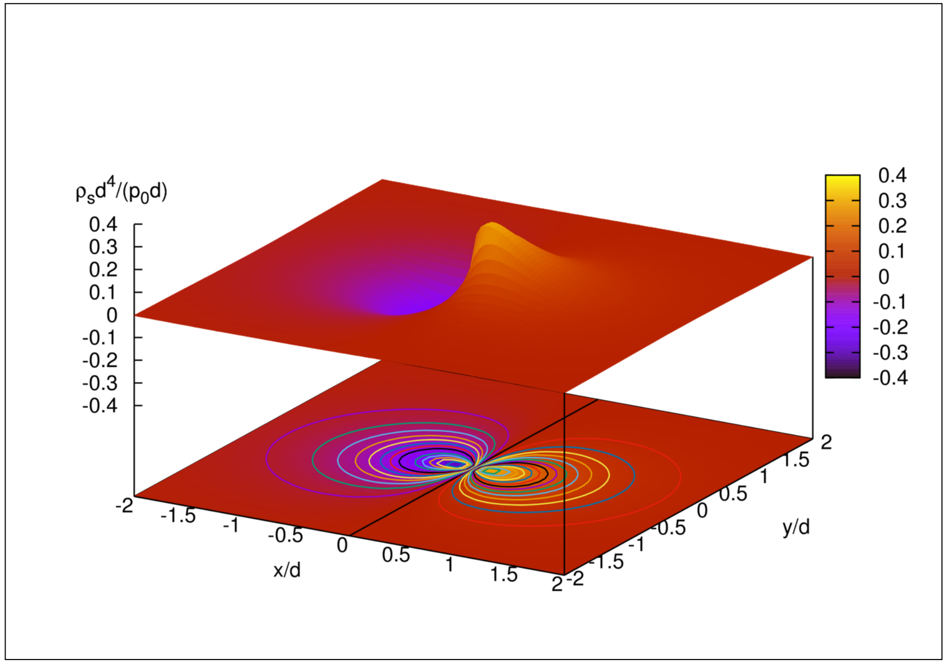
5

Víddarlaus stærð er

$$\frac{\rho_s d^4}{p_0 d} = -\frac{\cos\phi}{\pi\left(1+\frac{R^2}{d^2}\right)^2}$$

því $\quad [p_0] = LQ$

Mynd af hleðsludreifingunni er á næstu síðu. Heildarhleðsla yfirborðsins er 0, dreifingin er greinilega tvískautsdreifing. Tvískautið á yfirborðinu og í fjarlægð d frá því mynda fjórskaut.



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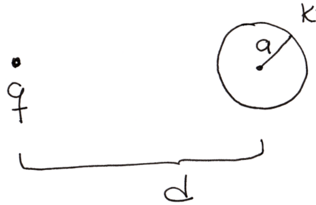


Kúla með hleðslu q
Kjörleiðari

Milli tveggja punkthleðslna q er alltaf fráhrindi kraftur
Hvað gerist hér?

Spegilhleðsla í kúlu: $\quad q_1 = -\frac{a}{d}q \quad$ í fjarlægð $\frac{a^2}{d}$ frá miðju

En kúlan var hlaðin með q, því setjum við í miðju kúlu $\quad q_2 = q - q_1$: hleðslan á kúlunni var q. q_2 er aukahleðslan sem þarf til að kúlan hafi hleðslu q eftir að frá er dregin yfirborðshleðslan sem q skautar á henni

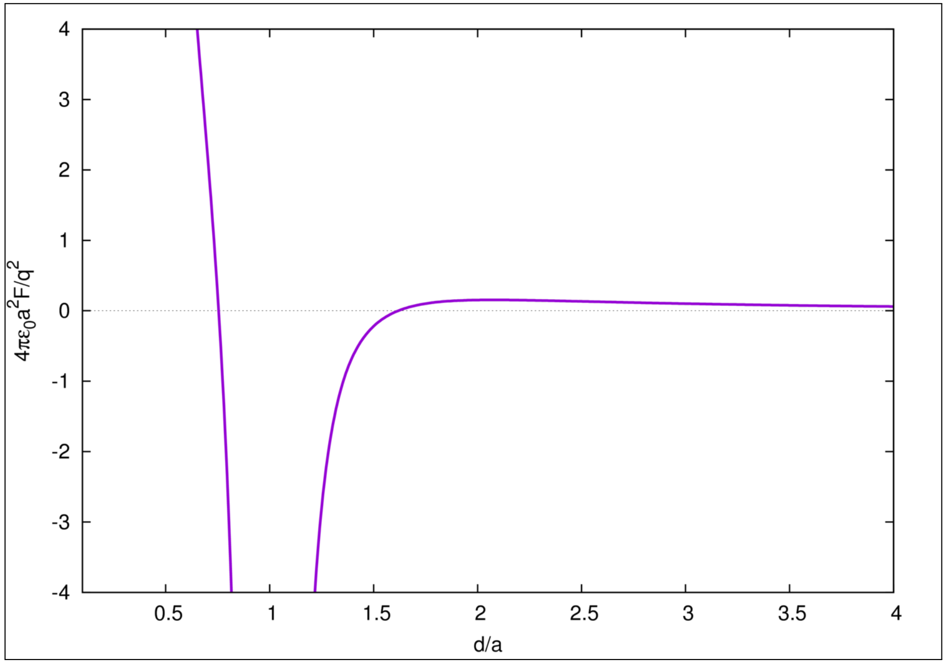
8

$$\to F = \frac{1}{4\pi\epsilon_0}\left\{\frac{q\cdot q_1}{\left(d-\frac{a^2}{d}\right)^2} + \frac{q\cdot q_2}{d^2}\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\left\{-\frac{q^2\frac{a}{d}}{\left(d-\frac{a^2}{d}\right)^2} + \frac{q\left(q+\frac{aq}{d}\right)}{d^2}\right\}$$

$$= \frac{q^2}{4\pi\epsilon_0}\left\{-\frac{ad}{\left(d^2-a^2\right)^2} + \frac{d^2+ad}{d^4}\right\}$$

Mynd á næstu síðu sýnir F(d), þar sést að F verður aðdráttarkraftur þegar $\frac{d}{a} \lesssim 1{,}61$ og að yfirborði kúlunnar



4
3
2
1
0
-1
-2
-3
-4
0.5
1
1.5
2
2.5
3
3.5
4
d/a
$4\pi\varepsilon_0 a^2 F/q^2$

Þunn einangrandi kúluskel með $\rho_s(\theta) = \rho_{s0}\cos(3\theta)$ (1)

(1) Hér er heildarhleðslan

$$Q_s = \oint ds\, \rho_s(\Omega) = 2\pi\rho_{s0}\int_0^{\pi} a^2 \sin\theta\, d\theta \cos(3\theta)$$

$$= 2\pi a^2 \rho_{s0}\int_0^{\pi} d\theta \left\{4\sin\theta\cos^3\theta - 3\cos\theta\sin\theta\right\}$$

$= 0$ hvort heildið sem reynt er í wxMaxima.

Yfirborðshleðslan er eins og skautunarhleðsla.

(2) Finnum V innan og utan kúlu (2)

þurfum að leysa jöfnu Poissons

$$\nabla^2 V(R,\theta) = -\frac{1}{\epsilon_0}\rho_s(R,\theta)$$

Utan og innan kúluskeljar þurfum við því að leysa jöfnu Laplace, skeytum síðan saman lausnunum með yfirborðshleðsluna í huga. Almenna lausnin er

$$V_n(R,\theta) = \left\{A_n R^n + B_n R^{-(n+1)}\right\} P_n(\cos\theta)$$

(3)

Innan kúlu er engin punkthleðsla $\to B_n = 0$ f. $\forall n$ þar

$$\to V_n^i(R,\theta) = A_n R^n P_n(\cos\theta)$$

Utan kúlu getur lausnin ekki vaxið án takmarkana $\to A_n$ f. $\forall n$ þar

$$V_n^o(R,\theta) = B R^{-(n+1)} P_n(\cos\theta)$$

Yfirborðshleðslan leiðir til brots í afleiðu V þ.a.

$$\hat{a}_{n_2}\cdot(\bar{D}_1 - \bar{D}_2) = \rho_s$$

(4)

Notum okkur hér $\bar{D} = \epsilon_0 \bar{E}$ (enginn rafsvari) og $\bar{E} = -\bar{\nabla} V$ ásamt $\hat{a}_{n_2} = \hat{a}_R \to$

$$(*) \quad \left\{\partial_R V^o(R,\theta) - \partial_R V^i(R,\theta)\right\}\Big|_{R=a} = -\frac{\rho_{s0}}{\epsilon_0}\cos(3\theta)$$

þar sem við höfum summað upp í almennu lausnirnar

$$\boxed{\begin{aligned} V^o(R,\theta) &= \sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta) \qquad R > a \\ V^i(R,\theta) &= \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) \qquad R < a \end{aligned}}$$

(5)

Við þurfum að umrita $\cos(3\theta)$ sem Legrende föll.

Einfaldast er að fletta upp Chebyshev fliðurliðum

$$\to T_3(\cos\theta) = \cos(3\theta) = \frac{1}{5}\left\{8P_3(\cos\theta) - 3P_1(\cos\theta)\right\}$$

Til að uppfylla jaðarskilyrðin (*) fáum við þá

$$-\sum_{n=0}^{\infty}(n+1)\frac{B_n}{a^{n+2}}P_n(\cos\theta) - \sum_{n=0}^{\infty} n A_n a^{n-1} P_n(\cos\theta)$$

$$= -\frac{\rho_{s0}}{\epsilon_0 5}\left\{8P_3(\cos\theta) - 3P_1(\cos\theta)\right\}$$

Legendre fliðurliðurnar eru stök í hornréttum grunni. Því einfaldast jaðarskilyrðin í

(6)

$$-2\frac{B_1}{a^3}P_1(\cos\theta) - A_1 P_1(\cos\theta) = \frac{3}{5}\frac{\rho_{s0}}{\epsilon_0}P_1(\cos\theta)$$

$$-4\frac{B_3}{a^5}P_3(\cos\theta) - 3A_3 a^2 P_3(\cos\theta) = -\frac{8}{5}\frac{\rho_{s0}}{\epsilon_0}P_3(\cos\theta)$$

og $B_n = 0$ og $A_n = 0$ ef $n \neq 1$ eða $n \neq 3$

$$-\frac{2}{a^3}B_1 - A_1 = \frac{3}{5}\frac{\rho_{s0}}{\epsilon_0}$$

$$-\frac{4}{a^5}B_3 - 3a^2A_3 = -\frac{8}{5}\frac{\rho_{s0}}{\epsilon_0}$$

(7)

Rafstöðumættið er samfellt í $R = a$

$$V^i(a^-) = V^o(a^+)$$

$$\to A_n a^n = B_n \frac{1}{a^{n+1}} \to B_n = A_n a^{2n+1}$$

því verða jaðarskilyrðin saman

$$-2A_1 - A_1 = \frac{3}{5}\frac{\rho_{s0}}{\epsilon_0} \longrightarrow A_1 = -\frac{\rho_{s0}}{5\epsilon_0}$$

$$-4a^2A_3 - 3a^2A_3 = -\frac{8}{5}\frac{\rho_{s0}}{\epsilon_0} \longrightarrow A_3 = \frac{8}{35}\frac{\rho_{s0}}{\epsilon_0 a^2}$$

$$\to B_1 = -\frac{\rho_{s0}}{5\epsilon_0}a^3, \qquad B_3 = \frac{8\rho_{s0}}{35\epsilon_0}a^5$$

(8)

$$V^o(R,\theta) = \frac{\rho_{s0}}{5\epsilon_0}\left[-\frac{a^3}{R^2}P_1(\cos\theta) + \frac{8}{7}\frac{a^5}{R^4}P_3(\cos\theta)\right]$$

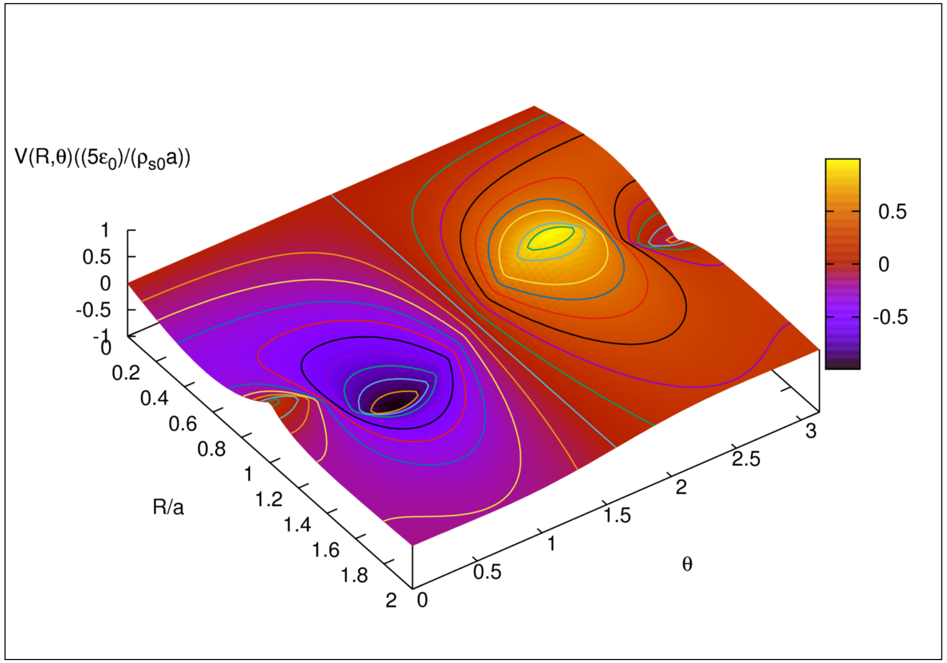
Hér má sjá bæði tvískauts og fjórskauts liði:

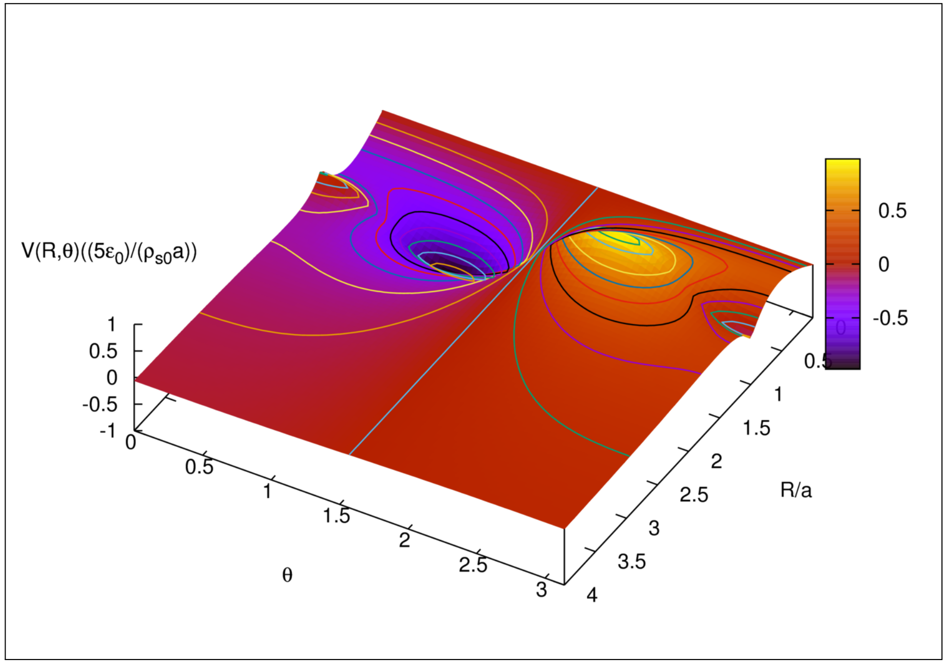
$$= \frac{\rho_{s0}a}{5\epsilon_0}\left[-\left(\frac{a}{R}\right)^2 P_1(\cos\theta) + \frac{8}{7}\left(\frac{a}{R}\right)^4 P_3(\cos\theta)\right]$$

$$V^i(R,\theta) = \frac{\rho_{s0}a}{5\epsilon_0}\left[-\left(\frac{R}{a}\right)P_1(\cos\theta) + \frac{8}{7}\left(\frac{R}{a}\right)^3 P_3(\cos\theta)\right]$$

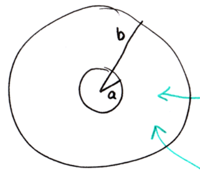
$$P_1(\cos\theta) = \cos\theta$$

$$P_3(\cos\theta) = \frac{1}{2}\left\{5\cos^3\theta - 3\cos\theta\right\}$$


V(R,θ)((5ε0)/(ρs0a))
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
R/a
0
0.5
1
1.5
2
2.5
3
θ
0.5
0
-0.5


V(R,θ)((5ε0)/(ρs0a))
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
2.5
3
θ
0.5
1
1.5
2
2.5
3
3.5
4
R/a
0.5
0
-0.5

1



Kjörleiðandi kúluskeljar

$$\epsilon(R) = \epsilon_0\left\{1+\left(\frac{b}{R}\right)^2\right\}$$

$$\sigma(R) = \sigma_0\left\{1+\left(\frac{b}{R}\right)^2\right\}$$

1 Finnum leiðni þéttisins

Hér er hvorki hægt að nota Laplace til að finna mættið fyrir rafsviðið né strauminn

$$\hookrightarrow \quad \bar{\nabla}\cdot\bar{J} = 0$$

sem er varðveisla straumsins í sístæðu ástandi.

1

Milli skeljanna gildir

$$\oint \bar{J}\cdot d\bar{s} = 0$$

inn eða út um skeljarnar kemur I

$$\rightarrow \bar{J} = \frac{I}{4\pi R^2}\,\hat{a}_R$$

Veljum stefnu frá innri skel a á ytri.

Ohms lögmál gefur okkur þá

$$\bar{E} = \frac{\bar{J}(R)}{\sigma(R)}$$

2

$$\bar{E} = \frac{\bar{J}(R)}{\sigma(R)} = \hat{a}_R\,\frac{I}{4\pi R^2\sigma_0\left\{1+\left(\frac{b}{R}\right)^2\right\}} = \frac{\hat{a}_R I}{4\pi\sigma_0\left\{R^2+b^2\right\}}$$

Næst getum við fundið spennunum skeljanna

$$V_a - V_b = -\int_b^a \bar{E}\cdot d\bar{l} = -\int_b^a \frac{I\,dR}{4\pi\sigma_0\left\{R^2+b^2\right\}}$$

$$= \frac{I}{4\pi\sigma_0}\,\frac{1}{b}\arctan\left(\frac{x}{b}\right)\Bigg|_a^b = \frac{I}{4\pi\sigma_0 b}\left\{\arctan(1)-\arctan\left(\frac{a}{b}\right)\right\}$$

$$= \frac{I}{4\pi\sigma_0 b}\left\{\frac{\pi}{4}-\arctan\left(\frac{a}{b}\right)\right\} > 0 \quad \text{ef } a<b$$

3

$$G = \frac{I}{V_a-V_b} = \frac{4\pi\sigma_0 b}{\left\{\frac{\pi}{4}-\arctan\left(\frac{a}{b}\right)\right\}}$$

leiðni þéttisins

2 Dreifing frjálsra hleðslna í þéttinum?

Bolhleðslur má finna frá $\bar{\nabla}\cdot\bar{D} = \rho$

$$\bar{D} = \epsilon(R)\bar{E} = \epsilon_0\left\{1+\left(\frac{b}{R}\right)^2\right\}\frac{\hat{a}_R I}{4\pi R^2\sigma_0\left\{1+\left(\frac{b}{R}\right)^2\right\}}$$

$$= \frac{\hat{a}_R I\epsilon_0}{4\pi R^2\sigma_0}$$

$$\rho(R) = \bar{\nabla}\cdot\bar{D} = \frac{1}{R^2}\frac{d}{dR}\left\{\frac{I\epsilon_0}{4\pi\sigma_0}\right\} = \underline{0}$$

hverfandi bolhleðsla

4

Á skeljunum í $R = b^-$ og a^+ safnast fyrir hleðslur sem viðhalda spennunni á þéttinum. Þessu er lýst með jaðarskilyrðunum

$$\hat{a}_{n_2}\cdot(\bar{D}_1-\bar{D}_2) = \rho_s$$

$\underline{R = a^+}$

$$\rho_s(a^+) = \epsilon(a)\bar{E}(a)\cdot\hat{a}_R = \frac{\epsilon_0\left\{1+\left(\frac{b}{a}\right)^2\right\}I}{4\pi\sigma_0 a^2\left\{1+\left(\frac{b}{a}\right)^2\right\}}$$

$$= \frac{\epsilon_0 I}{4\pi\sigma_0 a^2}$$

$\underline{R = b^-}$

$$\rho_s(b^-) = -\epsilon(b)\bar{E}(b)\cdot\hat{a}_R = -\frac{\epsilon_0 2I}{4\pi\sigma_0 b^2\, 2}$$

$$= -\frac{\epsilon_0 I}{4\pi\sigma_0 b^2}$$

5

því sést strax að heildar frjálsu hleðslurnar styttast út.

③ Dreifing skautunarhleðsla í rafsvaranum

$$\overline{D} = \epsilon_0 \overline{E} + \overline{P} \;\rightarrow\; \overline{P} = \overline{D} - \epsilon_0 \overline{E} = \left\{\epsilon(R) - \epsilon_0\right\} \overline{E}$$

$$\epsilon(R) - \epsilon_0 = \epsilon_0 \left(\frac{b}{R}\right)^2 , \quad \overline{E} = \frac{\hat{a}_R I}{4\pi\sigma_0 \left\{R^2 + b^2\right\}}$$

$$\rightarrow \overline{P} = \frac{\hat{a}_R \epsilon_0 I b^2}{4\pi\sigma_0 R^2 \left\{R^2 + b^2\right\}}$$

þá skautunarhleðslur eru

$$\rho_p = -\overline{\nabla} \cdot \overline{P} = -\frac{1}{R^2}\left[\frac{d}{dR} \frac{\epsilon_0 I b^2}{4\pi\sigma_0 \left\{R^2 + b^2\right\}}\right]$$

6

$$\rightarrow \rho_p = \frac{1}{R^2} \frac{\epsilon_0 I b^2}{4\pi\sigma_0} \frac{2R}{\left\{R^2 + b^2\right\}^2} = \frac{\epsilon_0 I b^2}{2\pi\sigma_0 R \left\{R^2 + b^2\right\}^2}$$

skautunarhleðsla á yfirborði rafsvarans

$R = a^+$

$$\rho_{ps}(a^+) = \overline{P}(a^+) \cdot \hat{a}_n = -\overline{P}(a) = -\frac{\epsilon_0 I b^2}{4\pi\sigma_0 a^2 (a^2 + b^2)}$$

$R = b^-$

$$\rho_{ps}(b^-) = P(b) = \frac{\epsilon_0 I}{4\pi\sigma_0 2 (b^2)} = \frac{\epsilon_0 I}{8\pi\sigma_0 b^2}$$

7

Við höfum séð að heildar frjálsa hleðslan í þéttinum styttist út, en hvernig er heildar skautaðu hleðslunni háttað?

$$Q(a^+) = S_a \rho_{ps}(a^+) = \frac{\epsilon_0 I b^2}{\sigma_0 (a^2 + b^2)}$$

$$Q(b^-) = S_b \rho_{ps}(b^-) = \frac{\epsilon_0 I}{2\sigma_0}$$

$$Q_b = \int_a^b dv\, \rho_p = \frac{2\epsilon_0 I b^2}{\sigma_0} \int_a^b \frac{R\,dR}{\left\{R^2 + b^2\right\}^2}$$

8

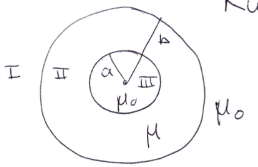
$$\rightarrow Q_b = \frac{\epsilon_0 I}{\sigma_0} \left\{\frac{b^2}{b^2 + a^2} - \frac{1}{2}\right\}$$

því sést greinilega að $Q_b + Q(a^+) + Q(b^-) = 0$

skautunarhleðslan styttist út í þéttinum.

1

Kúlustel í ytra segulsviði $\bar B = B_0 \hat a_z$



Engir frjálsir straumar $\to$ $\nabla\times\bar H = 0$

Engin einskaut $\to$ $\bar\nabla\cdot\bar B = 0$

$\to$ til er skalarmætti ϕ_m þ.a. $\bar H = -\nabla\phi_m$

og $\nabla^2\phi_m = 0$ (Gerum ráð fyrir að $\bar H = \frac{1}{\mu}\bar B$)

Jaðarskilyrði: fyrir $R >> b$ verður að gilda að

$$\bar B = B_0\hat a_z \to \phi_m(R,\theta) = -\frac{B_0}{\mu_0}R\cos\theta$$

eins verður að gilda fyrir báði yfirborðin (þ. $R = a$ og b)

$$\hat a_{n_2}\times(\bar H_1 - \bar H_2) = \bar J_s$$

$\bar B_n$ er samfellt í yfirborði

2

Yfirborð $R = b$

$$\hat a_R\times(\bar H_I - \bar H_{II}) = 0$$ ← engir frjálsir yfirb. straumar

$$\phi_m = \phi_m(R,\theta) \qquad \bar H = -\nabla\phi_m(R,\theta) = -\hat a_R\frac{\partial\phi_m}{\partial R} - \hat a_\theta\frac{1}{R}\frac{\partial\phi_m}{\partial\theta}$$

$$\to -\frac{\hat a_\phi}{R}\left\{\partial_\theta\phi_m^I(b^+,\theta) - \partial_\theta\phi_m^{II}(b^-,\theta)\right\} = 0$$

$$\to \partial_\theta\phi_m^I(b,\theta) = \partial_\theta\phi_m^{II}(b,\theta) \quad (1)$$

auk

$$-\mu_0\partial_R\phi_m^I(b,\theta) = -\mu\partial_R\phi_m^{II}(b,\theta) \quad (2)$$ ← B_n samfellt

Yfirborð $R = a$

$$-\hat a_R\times(\bar H_{III} - \bar H_{II}) = 0$$

$$\to \partial_\theta\phi_m^{III}(a,\theta) \underset{(3)}{=} \partial_\theta\phi_m^{II}(a,\theta), \text{ og } -\mu_0\partial_R\phi_m^{III}(a,\theta) \underset{(4)}{=} -\mu\partial_R\phi_m^{II}(a,\theta)$$

3

Í kúluhnitum er almenna lausn Laplace

$$\phi_m(R,\theta) = \sum_{n=0}^{\infty}\left\{A_nR^n + B_nR^{-(n+1)}\right\}P_n(\cos\theta)$$

fyrir $R > b$ verður að fella formið $\phi_m^I(R,\theta) = -H_0R\cos\theta$ að ráða þ. R verður stórt m.v. b

$$\to \phi_m^I(R,\theta) = -H_0R\cos\theta + \sum_{n=0}^{\infty}B_n^I R^{-(n+1)}P_n(\cos\theta)$$

Fyrir $a < R < b$

$$\phi_m^{II}(R,\theta) = \sum_{n=0}^{\infty}\left\{A_n^{II}R^n + \frac{B_n^{II}}{R^{n+1}}\right\}P_n(\cos\theta)$$

4

Fyrir $R < a$, Enginn sérstöðupunktar

$$\to \phi_m^{III}(R,\theta) = \sum_{n=0}^{\infty}A_n^{III}R^nP_n(\cos\theta)$$

Eins og fyrir rafsvarandi kúlu í ytra rafsviði kemur í ljós að aðeins stuðlar með $n = 1$ eru mögulegir

$$\to \phi_m^I(R,\theta) = -H_0R\cos\theta + \frac{B_1^I\cos\theta}{R^2}$$

$$\phi_m^{II}(R,\theta) = \left\{A_1^{II}R + \frac{B_1^{II}}{R^2}\right\}\cos\theta$$

$$\phi_m^{III}(R,\theta) = A_1^{III}R\cos\theta$$

(5)

① -> $$-H_0 b + \frac{B_1^{I}}{b^2} = A_1^{II} b + \frac{B_1^{II}}{b^2}$$

② -> $$-\mu_0\left\{-H_0 - \frac{2B_1^{I}}{b^3}\right\} = -\mu\left\{A_1^{II} - \frac{2B_1^{II}}{b^3}\right\}$$

③ -> $$A_1^{III} a = A_1^{II} a + \frac{B_1^{II}}{a^2}$$

④ -> $$-\mu_0\left\{A_1^{III}\right\} = -\mu\left\{A_1^{II} - \frac{2B_1^{II}}{a^3}\right\}$$

---

① -> $$B_1^{I} - A_1^{II} b^2 - B_1^{II} = b^3 H_0$$

② -> $$2B_1^{I}\mu_0 + \mu A_1^{II} b^3 - 2\mu B_1^{II} = -b^3 H_0 \mu_0$$

(6)

③ -> $$A_1^{II} a^3 + B_1^{II} - A_1^{III} a^3 = 0$$

④ -> $$\mu A_1^{II} a^3 - 2\mu B_1^{II} - \mu_0 A_1^{III} a^3 = 0$$

---

4 jöfnur B_1^{I}, B_1^{II}, A_1^{II} og A_1^{III} óþekktir fastar

$$\begin{pmatrix} 1 & -1 & -b^3 & 0 \\ 2\mu_0 & -2\mu & \mu b^3 & 0 \\ 0 & 1 & a^3 & -a^3 \\ 0 & -2\mu & \mu a^3 & -\mu_0 a^3 \end{pmatrix} \begin{pmatrix} B_1^{I} \\ B_1^{II} \\ A_1^{II} \\ A_1^{III} \end{pmatrix} = \begin{pmatrix} b^3 H_0 \\ -\mu_0 b^3 H_0 \\ 0 \\ 0 \end{pmatrix}$$

(7)

Deilum með μ_0 í gegnum jöfnu ② og ③ og skilgreinum $\mu_r = \frac{\mu}{\mu_0}$

$$\begin{pmatrix} 1 & -1 & -b^3 & 0 \\ 2 & -2\mu_r & \mu_r b^3 & 0 \\ 0 & 1 & a^3 & -a^3 \\ 0 & -2\mu_r & \mu_r a^3 & -a^3 \end{pmatrix} \begin{pmatrix} B_1^{I} \\ B_1^{II} \\ A_1^{II} \\ A_1^{III} \end{pmatrix} = \begin{pmatrix} b^3 H_0 \\ -b^3 H_0 \\ 0 \\ 0 \end{pmatrix}$$

með lausn

(8)

$$B_1^{I} = \frac{\left\{b^3 a^3(-2\mu_r^2 + \mu_r + 1) + b^6(2\mu_r^2 - \mu_r - 1)\right\} H_0}{b^3(2\mu_r^2 + 5\mu_r + 2) - 2a^3\mu_r^2 + 4a^3\mu_r - 2a^3}$$

$$= \frac{-a^3(2\mu_r^2 - \mu_r - 1) + b^3(2\mu_r^2 - \mu_r - 1)}{(2\mu_r^2 + 5\mu_r + 2) + \frac{a^3}{b^3}(-2\mu_r^2 + 4\mu_r - 2)} H_0$$

$$= \left\{\frac{(2\mu_r + 1)(\mu_r - 1)}{(2\mu_r + 1)(\mu_r + 2) - 2\left(\frac{a}{b}\right)^3(\mu_r - 1)^2}\right\}(b^3 - a^3) H_0$$

---

og

$$A_1^{III} = -\left\{\frac{9\mu_r}{(2\mu_r + 1)(\mu_r + 2) - 2\left(\frac{a}{b}\right)^3(\mu_r - 1)^2}\right\} H_0$$

(9)

Utan kúluskeljar var $\bar{B} = B_0 \hat{a}_z$ fast svið

kúluskelin bætir við tvístautssviði fyrir utan kúlust.

Í réttu hlutfalli við B_1^{I}

Innan kúluskeljar myndast

$$\phi_m^{III} = A_1^{III} \, r \cos\theta = A_1^{III} z$$

$$\bar{H}^{III} = -\bar{\nabla}\phi_m^{III} = -A_1^{III} \hat{a}_z \quad \text{fast segulsvið!}$$

$$B^{III} = -\mu_0 A_1^{III} \hat{a}_z = \frac{9 \mu H_0}{(2\mu_r+1)(\mu_r+2) - 2\left(\frac{a}{b}\right)^3 (\mu_r-1)^2}$$

$$= \frac{9\mu_r B_0}{\mu_r^2 \left\{ \left(2+\frac{1}{\mu_r}\right)\left(1+\frac{2}{\mu_r}\right) - 2\left(\frac{a}{b}\right)^3\left(1-\frac{1}{\mu_r}\right)^2 \right\}}$$

(10)

$$\rightarrow B^{III} \rightarrow 0 \quad \text{þegar} \quad \mu_r = \frac{\mu}{\mu_0} \rightarrow \infty$$

fyrir járnseglandi efni er $\mu_r \sim 10^3 - 10^6$

Við sjáum því segulskýlingu

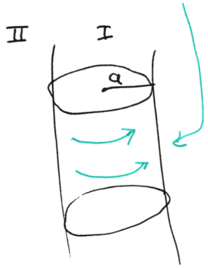
---

fyrir sterka andseglun fæst $\mu = 0$, $\mu_r = 0$

$$B^{III} = \frac{9\mu_r B_0}{(2\mu_r+1)(\mu_r+2) - 2\left(\frac{a}{b}\right)^3 (\mu_r-1)^2} \rightarrow 0$$

(1) Spóla, þunn, með geisla a og straumþéttleika (1)

$\bar{J} = J_0 \hat{a}_\phi$ þegar $r=a$



Óendanleg -> búumst að eins við að $\bar{A}$ hafi ϕ-hnit, þurfum að leysa $\nabla^2 \bar{A} = -\mu_0 \bar{J}$ eða til að vera örugg

$$\boxed{\bar{\nabla}\times\bar{\nabla}\times\bar{A} = \mu_0 \bar{J}}$$

$$\bar{\nabla}\times\bar{A} = \hat{a}_z \left\{ \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) \right\}$$

$$\rightarrow \bar{\nabla}\times\bar{\nabla}\times\bar{A} = -\hat{a}_\phi \left\{ \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) \right] \right\}$$

$$= -\hat{a}_\phi \left\{ \frac{\partial}{\partial r}\left[\frac{1}{r}\left(r\frac{\partial}{\partial r}A_\phi + A_\phi \right) \right] \right\}$$

(2)

$$\rightarrow \bar{\nabla}\times\bar{\nabla}\times\bar{A} = -\hat{a}_\phi \left\{ \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}A_\phi + \frac{A_\phi}{r} \right) \right\}$$

$$= -\hat{a}_\phi \left\{ \frac{\partial^2}{\partial r^2}A_\phi + \frac{1}{r}\frac{\partial}{\partial r}A_\phi - \frac{A_\phi}{r^2} \right\}$$

Því er jafnan sem við verðum að leysa

$$\boxed{\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} \right\} A_\phi = 0 \quad \text{ef } r \neq a}$$

Engir þættir $\bar{B}$ sem þvert á spólu -> jaðarskilyrðin eru

$$\hat{a}_{n_2} \times (\bar{H}_1 - \bar{H}_2) = \bar{J}_s$$

setjum $\bar{B} = \mu_0 \bar{H}$ og veljum

$2 \leftrightarrow \text{II}$

$1 \leftrightarrow \text{I}$

(3)

Þá fæst

$$-\hat{a}_r \times \hat{a}_z \left\{ \left[\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi^{\text{I}}) \right] - \left[\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi^{\text{II}}) \right] \right\} = \mu_0 J_0 \hat{a}_\phi$$

$$\rightarrow \left\{ \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi^{\text{I}}) \right\} - \left\{ \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi^{\text{II}}) \right\} = \mu_0 J_0$$

Reynum lausn $A_\phi(r) = C r^n$ í

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} \right\} A_\phi = 0$$

$$\rightarrow n(n-1) r^{n-2} + n r^{n-2} - r^{n-2} = 0$$

$$\rightarrow n(n-1) + n - 1 = 0 \quad \text{eða} \quad n^2 - n + n - 1 = 0$$

$$\rightarrow n = \pm 1$$

(4)

Lausn á svæði I, innan spólu er þá

$$A_\phi^{\text{I}} = C_1 r \qquad r < a$$

því þar getur ekki verið sérstöðu-punktur í $r = 0$

Á svæði II er lausn

$$A_\phi^{\text{II}} = C_2 \frac{1}{r} \qquad r > a$$

Jaðarskilyrðið gefur þá

$$2C_1 - 0 = \mu_0 J_0$$

Því er

$$\boxed{A_\phi^{\text{I}}(r) = \frac{\mu_0 J_0}{2} r}$$

Það gefur

$$\bar{B} = \bar{\nabla}\times\bar{A}$$

þ.a.

$$\bar{B}^{\text{I}} = \hat{a}_z \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 J_0}{2} r^2 \right)$$

$$= \hat{a}_z \mu_0 J_0$$

Fæst segulflæðisvið eins og við bjuggumst við

Ytri lausnin er enn

$$A_\phi^{II}(r) = \frac{C_2}{r}$$

sem gefur $\bar{B}^{II} = 0$

eins og búist var við, en getum við fundið C_2?

Við vitum að $\bar{B} = \bar{\nabla}\times\bar{A}$

$$\to \underbrace{\oint_S \bar{B}\cdot d\bar{s}}_{\Phi_B} = \oint_S (\bar{\nabla}\times\bar{A})\cdot d\bar{s}$$

$$= \oint_C d\bar{l}\cdot\bar{A}$$

Á svæði II, utan spólu, er $\Phi_B = \mu_0 J_0 \pi a^2$

$$\Phi_B = \oint d\bar{l}\cdot\bar{A}$$

$$\mu J_0 \pi a^2 = 2\pi r\, C_2 \frac{1}{r}$$

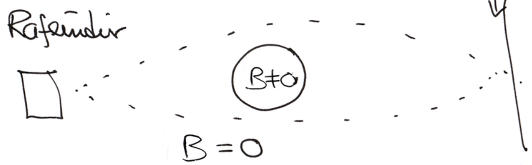
$$\to C_2 = \frac{\mu J_0 a^2}{2}$$

$$\to \boxed{A_\phi^{II} = \frac{\mu J_0 a^2}{2r}}$$

því sést líka að

$$A_\phi^{I}(a^-) = A_\phi^{II}(a^+)$$

Tengsl við hrif Ahronovs og Bohms séð ofan á spólu



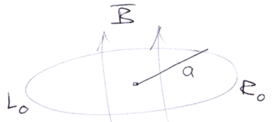
mynstur, víxlmynstur rafeinda bylgna er háð B þó svo rafeindirnar fari aldrei um svæði með $\bar{B} \neq 0$

$\bar{A} \neq 0$ utan spólu, $\bar{A}$ er í jöfnu Schrödingers ásamt V (ekki $\bar{E}$ og $\bar{B}$)

Lykkja með a, L_0 og R_0 og segulflæði $\Phi(t) = \Phi_0 e^{-\Gamma t}\sin(\omega t)$ (1)

finnum strauminn sem spanast í henni.

Lögmál Faradays



$$\oint_c \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}$$

$$\frac{d\Phi}{dt} = \Phi_0 e^{-\Gamma t}\left\{\omega\cos(\omega t) - \Gamma\sin(\omega t)\right\} + L_0\frac{d}{dt}i(t)$$ því

heildar flæðið um lykkjuna er $\Phi(t) = \Phi_0 e^{-\Gamma t}\sin(\omega t) + L_0 i(t)$

$$\to \quad R_0 i(t) = -\Phi_0 e^{-\Gamma t}\left\{\omega\cos(\omega t) - \Gamma\sin(\omega t)\right\} - L_0\frac{di(t)}{dt}$$

$$\to \quad L_0\frac{di(t)}{dt} + R_0 i(t) = -\Phi_0 e^{-\Gamma t}\left\{\omega\cos(\omega t) - \Gamma\sin(\omega t)\right\}$$

(2)

með upphafsgildi $i(0) = 0$

Fyrir 1. stigs jöfnuna $y' + p(t)y = q(t)$ er almenna lausnin

$$y(t) = y(0)e^{-P(t)} + e^{-P(t)}\int_0^t ds\, e^{P(s)}q(s)$$

$$P(t) = \int_0^t ds\, p(s)$$

ef upphafspunkturinn er $t = 0$

$$P(t) = \int_0^t ds\,\frac{R_0}{L_0} = \frac{R_0}{L_0}t$$

því fæst

$$i(t) = -e^{-\frac{R_0 t}{L_0}}\frac{\Phi_0}{L_0}\int_0^t ds\, e^{\frac{R_0}{L_0}s}e^{-\Gamma s}\left\{\omega\cos(\omega s) - \Gamma\sin(\omega s)\right\}$$

(3)

Setjum $\alpha = \frac{R_0}{L_0}$

$$i(t) = -e^{-\alpha t}\frac{\Phi_0}{L_0}\int_0^t ds\, e^{(\alpha-\Gamma)s}\left\{\omega\cos(\omega s) - \Gamma\sin(\omega s)\right\}$$

$$= \frac{e^{-\alpha t}\Phi_0}{L_0}\left[\frac{\alpha\omega}{\omega^2 + (\alpha-\Gamma)^2} - \frac{e^{(\alpha-\Gamma)t}\left\{(\omega^2 + (\Gamma^2 - \alpha\Gamma))\sin(\omega t) + \alpha\omega\cos(\omega t)\right\}}{\omega^2 + (\alpha-\Gamma)^2}\right]$$

$$= \frac{e^{-\alpha t}\Phi_0}{L_0\left\{\omega^2 + (\alpha-\Gamma)^2\right\}}\left[\alpha\omega - e^{(\alpha-\Gamma)t}\left\{(\omega^2 + \Gamma(\Gamma-\alpha))\sin(\omega t) + \alpha\omega\cos(\omega t)\right\}\right]$$

augljóslega sést að $i(0) = 0$

(4)

Eðlilegt er því ~~að~~ ~~skoða~~ myndir af

$$\frac{i(t)\omega^2\Phi_0}{L_0} = \frac{e^{-\frac{\alpha}{\omega}(t\omega)}}{1 + \left(\frac{\alpha-\Gamma}{\omega}\right)^2}\left[\frac{\alpha}{\omega} - e^{\frac{(\alpha-\Gamma)}{\omega}t\omega}\left\{\left(1 + \frac{\Gamma}{\omega}\left(\frac{\Gamma-\alpha}{\omega}\right)\right)\sin(\omega t) + \frac{\alpha}{\omega}\cos(\omega t)\right\}\right]$$

$$\mathcal{E} = -\frac{d\Phi}{dt} = \underbrace{-\Phi_0 e^{-\Gamma t}\left\{\omega\cos(\omega t) - \Gamma\sin(\omega t)\right\}}_{\text{v. breyting á ytra flæði}} \underbrace{- L_0\frac{d}{dt}i(t)}_{\text{v. breytingar á straumi í rás}}$$

$$= R_0 i(t)$$

(5)

$$i(t) = \frac{\Phi_0}{L_0\left\{\omega^2+(\alpha-\Gamma)^2\right\}}\left[\alpha\omega e^{-\alpha t} - e^{-\Gamma t}\left\{\left(\omega^2+\Gamma(\Gamma-\alpha)\right)\sin(\omega t)+\alpha\omega\cos(\omega t)\right\}\right]$$

$$Q = \int_0^\infty dt\, i(t) = \frac{\Phi_0}{L_0\left\{\omega^2+(\alpha-\Gamma)^2\right\}}\left[\omega - \frac{\omega\left(\omega^2+\Gamma(\Gamma-\alpha)\right)}{\omega^2+\Gamma^2} - \frac{\Gamma\alpha\omega}{\omega^2+\Gamma^2}\right]$$

$$= 0$$

Óháð stikunum, Γ, ω, $\alpha = \frac{R_0}{L_0}$ og Φ_0 er heildar hleðslu-flutningurinn alltaf 0, jafnvel þó $\frac{\Gamma}{\omega} = 1.0$ þ.a. Φ-púlsinn sé mest í aðra áttina. $i(t)$ er sett saman úr $-\frac{d\Phi}{dt}$ og $-L_0\frac{d}{dt}i(t)$

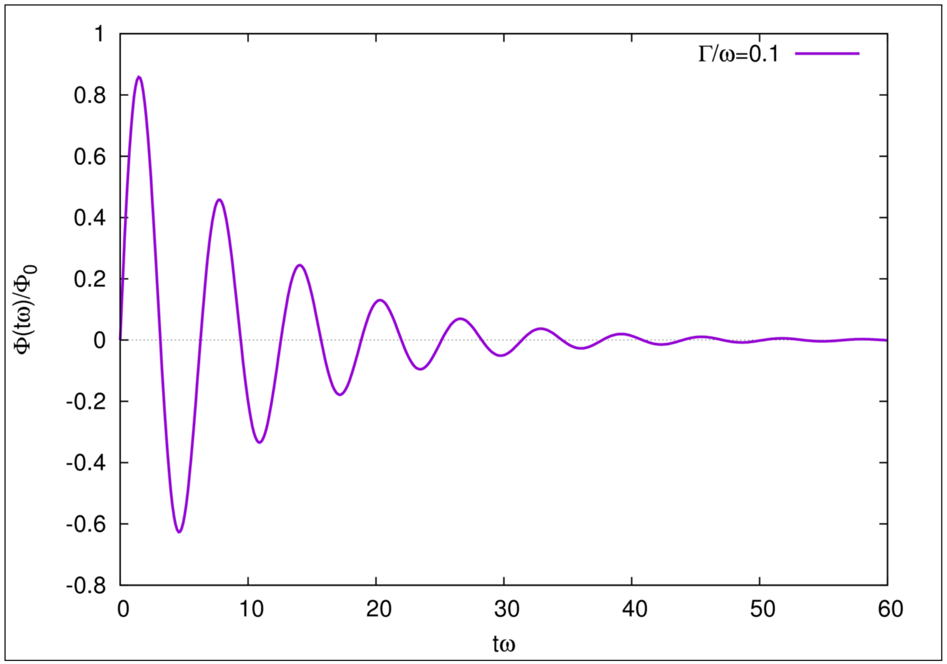
(6)

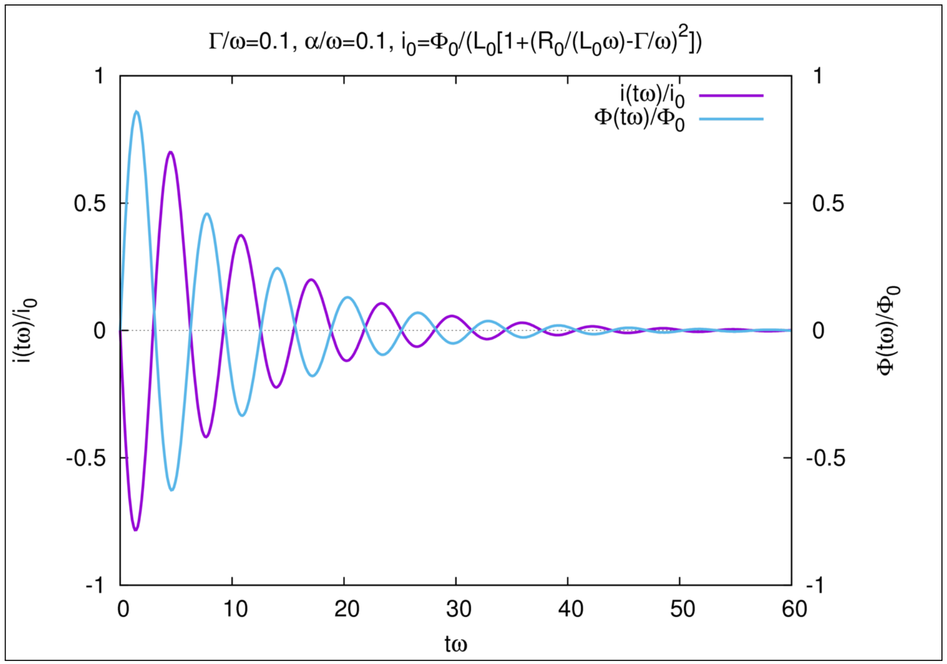
Tíma skala setja breyturnar Γ, ω og $\alpha = \frac{R_0}{L_0}$

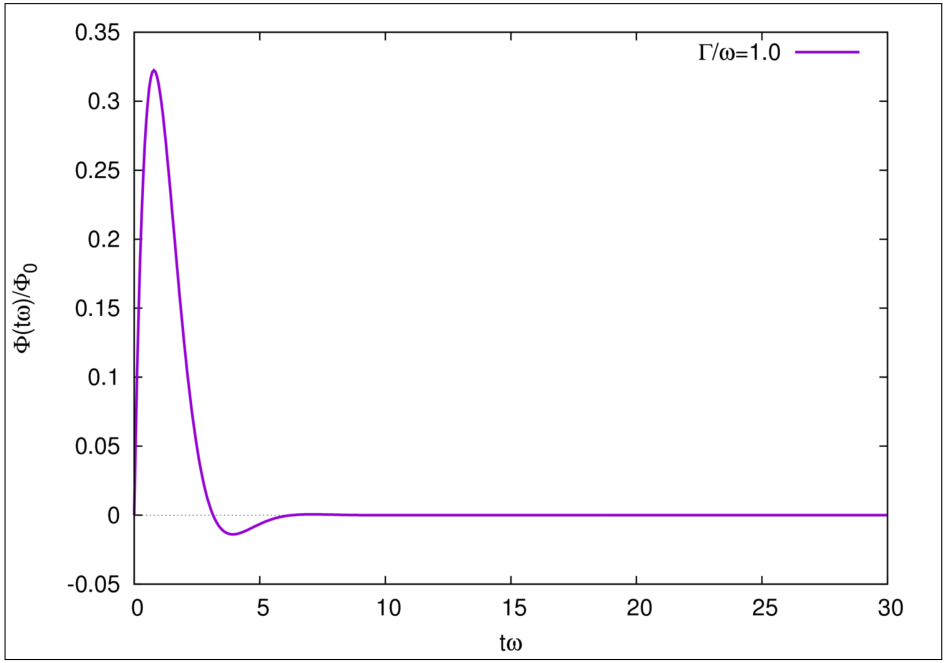
α : Setur náttúrulegan tímaskala spólunnar

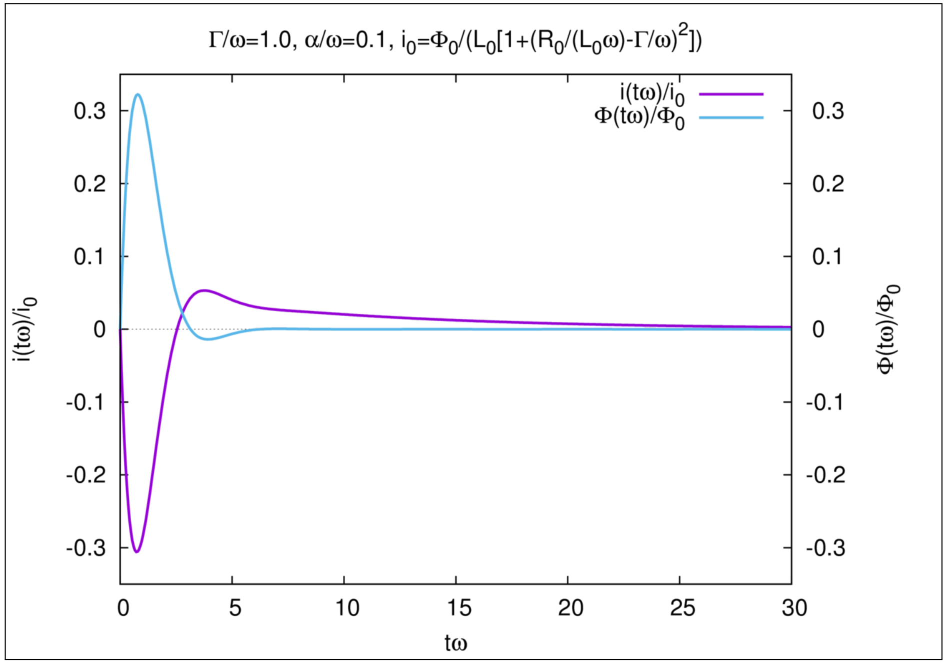
Γ og ω : Stýra tímahegðun flæðispúls, ytri.

Í $-\frac{d\Phi}{dt}$ ráða $\frac{\Gamma}{\omega}$, en í $-L_0\frac{d}{dt}i(t)$ kemur líka fyrir $\alpha = \frac{R_0}{L_0}$









(11)

Reynum aðra tegund lausnar (Aute, tilgamans)

$$\frac{di(t)}{dt} + \frac{R_0}{L_0} i(t) = -\frac{d\Phi^{ext}(t)}{L_0 dt} = -\frac{\Phi_0 e^{-\Gamma t}}{L_0}\{\omega\cos(\omega t) - \Gamma\sin(\omega t)\}$$

þar sem
$$\Phi^{ext}(t) = \Phi_0 e^{-\Gamma t}\sin(\omega t)$$

er ytra flæðið sem breytist í tíma. Heildarflæðið er því
$$\Phi(t) = \Phi^{ext}(t) + L_0 i(t)$$

eins og við höfum séð. Köllum $\alpha = \frac{R_0}{L_0}$

(12)

$$\frac{di(t)}{dt} + \alpha i(t) = -\frac{1}{L_0}\frac{d\Phi^{ext}(t)}{dt}$$

$$\rightarrow \int_0^t ds\,\frac{di(s)}{ds} + \alpha\int_0^t ds\, i(s) = -\frac{1}{L_0}\int_0^t ds\,\frac{d\Phi^{ext}(s)}{ds}$$

$$\rightarrow \quad i(t) - i(0) + \alpha\int_0^t ds\, i(s) = -\frac{1}{L_0}\left\{\Phi^{ext}(t) - \Phi^{ext}(0)\right\}$$

Gefum okkur að $i(0) = 0$ og $\Phi^{ext}(0) = 0$

$$\rightarrow \quad i(t) = -\frac{1}{L_0}\Phi^{ext}(t) - \alpha\int_0^t ds\, i(s)$$

Tíma afleiða þessarar jöfnu gefur afleiðujöfnuna sem við höfðum áður

(13)

Heildisjafnan

$$\boxed{i(t) = -\frac{1}{L_0}\Phi^{ext}(t) - \alpha\int_0^t ds\; i(s)} \qquad (*)$$

inniheldur jaðarstigyrðin. Hon nevnist jafna Volterra af annari tegund, við einfaldari kjarnanum $K(s) = -\alpha$.
Út úr henni lesum vit:

* Áður viðnáms, $R_0=0$ ($\alpha=0$), myndast streymur sum eyðir ytra flóðinum nákvæmliga $i(t) = -\frac{1}{L_0}\Phi^{ext}(t)$

* Ef $\alpha \neq 0$, tá er $i(t)$ háð forsøgustreymsins um kervið (Hundurin bítur støðugt í róvuna á sær!)

(14)

Loysn (*) samkvæmt Handbook of Integral Equations, eftir A.D. Polyanin og A.V. Manzhirov (2.1.1)

$$i(t) = -\frac{1}{L_0}\Phi^{ext}(t) + \alpha\int_0^t e^{-\alpha(t-s)}\frac{1}{L_0}\Phi^{ext}(s),$$

sum er í samræmi við almennu loysnina sum vit fundum fyrir afleiddu jøfnuna ef vit reyna innsetingu á $\Phi^{ext}(t)$, (men ekki sama form).

Skoðum aftur (*)

$$i(t) + \alpha\int_0^t ds\; i(s) = -\frac{1}{L_0}\Phi^{ext}(t)$$

Skrivum hetta tøkniliga sum

$$\boxed{\{1+\alpha K\}\, i = F}$$

(15)

1 : Eindarvirki
K : heildisvirki
F : pettur virki

— umbeytum í fylki → . . . .

i: virðið sum vit viljum finna

$$\rightarrow \boxed{i = \{1+\alpha K\}^{-1}\cdot F}$$

Loysnin er formliga røð í α, hvør liður við α^n stendur fyrir n-summing á hundinum at bíta í róvuna

Hetta sæst ekki á afleiddu jøfnuni, men í loysnini er $e^{-\alpha t}$ til komið vegna sjálvendurtøkuns bíts ella summings

(16)

Tí eru hesar upplýsingar bíta á afleiddu jøfnuni og jaðarstigyrðunum, men tær sjást betur í heildisjøfnuni.

Við hundinum at bíta í stertin á sær er her alt við spurningina hvørt neyðugt sæst at bíta við ytra Φ flóðinum sum sjálvspanið myndar, ella hvørt taka tørvur til lít til tess at tað flóði spani líka streym og tanning, koll at kolli út í $\rightarrow \infty$.

1

Flöt rafsegulbylgja með bylgjulengd λ fellur á spegil:

$$\beta_0 = \frac{2\pi}{\lambda}$$

$$\epsilon_r = 3.0$$

$$\mu_r = 1.0$$

Veljum

$$\bar{E}_1 = \hat{a}_x \left\{ E_{io} e^{i\beta_0 z} + E_{ro} e^{-i\beta_0 z} \right\}$$

$$\bar{H}_1 = \hat{a}_y \frac{1}{\eta_0} \left\{ E_{io} e^{i\beta_0 z} - E_{ro} e^{-i\beta_0 z} \right\}$$

$$\bar{E}_2 = \hat{a}_x \left\{ E_2^+ e^{i\beta_2 z} + E_2^- e^{-i\beta_2 z} \right\}$$

$$\bar{H}_2 = a_y \frac{1}{\eta_2} \left\{ E_2^+ e^{ik_2 z} - E_2^- e^{-i\beta_2 z} \right\}$$

2

Engin segulvirkni í glerplötunni, $\mu_r = 1$

því eru jaðarskilyrðin

$$E_{1t} = E_{2t} \quad \text{og} \quad \hat{a}_{n_2} \times (\bar{H}_1 - \bar{H}_2) = \bar{J}_s$$

Í $z = -d$ þýðir það

$$\bar{E}_1(-d) = \bar{E}_2(-d)$$

$$\bar{H}_1(-d) = \bar{H}_2(-d) \leftarrow$$ því enginn yfirborðsstraumur myndast í $z = -d$

Í $z = 0$ gildir

$$\bar{E}_2(0) = 0 \leftarrow$$ því $E = 0$ inni í leiðara

$$-\hat{a}_z \times \bar{H}_2 = \bar{J}_s \leftarrow$$ er yfirborðsstraumur á leiðara

3

Í $z = -d$

$$E_{io} e^{-i\beta_0 d} + E_{ro} e^{i\beta_0 d} = E_2^+ e^{-i\beta_2 d} + E_2^- e^{i\beta_2 d}$$

$$\frac{E_{io} e^{-i\beta_0 d} - E_{ro} e^{i\beta_0 d}}{\eta_0} = \frac{E_2^+ e^{-i\beta_2 d} - E_2^- e^{i\beta_2 d}}{\eta_2}$$

Í $z = 0$

$$E_2^+ + E_2^- = 0$$, hugsum okkur E_{io} gefið

$$\rightarrow \quad E_{io} e^{-i\beta_0 d} + E_{ro} e^{i\beta_0 d} = E_2^+ \left\{ e^{-i\beta_2 d} - e^{i\beta_2 d} \right\}$$

$$\eta_2 \left\{ E_{io} e^{-i\beta_0 d} - E_{ro} e^{i\beta_0 d} \right\} = \eta_0 E_2^+ \left\{ e^{-i\beta_2 d} + e^{i\beta_2 d} \right\}$$

4

$$E_{ro} e^{i\beta_0 d} - E_2^+ \left\{ e^{-i\beta_2 d} - e^{i\beta_2 d} \right\} = -E_{io} e^{-i\beta_0 d}$$

$$-\eta_2 E_{ro} e^{i\beta_0 d} - \eta_0 E_2^+ \left\{ e^{-i\beta_2 d} + e^{i\beta_2 d} \right\} = -\eta_2 E_{io} e^{-i\beta_0 d}$$

$$\begin{pmatrix} e^{i\beta_0 d} & -\left\{ e^{-i\beta_2 d} - e^{i\beta_2 d} \right\} \\ -\eta_2 e^{i\beta_0 d} & -\eta_0 \left\{ e^{-i\beta_2 d} + e^{i\beta_2 d} \right\} \end{pmatrix} \begin{pmatrix} E_{ro} \\ E_2^+ \end{pmatrix} = \begin{pmatrix} -E_{io} e^{-i\beta_0 d} \\ -\eta_2 E_{io} e^{-i\beta_0 d} \end{pmatrix}$$

(5)

$$E_{ro} = \frac{\eta_0 e^{-i\beta_0 d}\left\{e^{-i\beta_2 d}+e^{i\beta_2 d}\right\} - \eta_2 e^{-i\beta_0 d}\left\{e^{-i\beta_2 d}-e^{i\beta_2 d}\right\}}{\eta_2 e^{i\beta_0 d}\left\{e^{-i\beta_2 d}-e^{i\beta_2 d}\right\} + \eta_0 e^{i\beta_0 d}\left\{e^{-i\beta_2 d}+e^{i\beta_2 d}\right\}} E_{io}$$

$$E_2^+ = \frac{2\eta_2}{\eta_2 e^{i\beta_0 d}\left\{e^{-i\beta_2 d}-e^{i\beta_2 d}\right\} + \eta_0 e^{i\beta_0 d}\left\{e^{-i\beta_2 d}+e^{i\beta_2 d}\right\}} E_{io}$$

Umritum

$$E_{ro} = \frac{e^{-2i\beta_0 d}\left\{\eta_0 \cos(\beta_2 d) + \eta_2 i \sin(\beta_2 d)\right\}}{\eta_0 \cos(\beta_2 d) - \eta_2 i \sin(\beta_2 d)} E_{io}$$

$$\beta_0 = \frac{2\pi}{\lambda} \qquad \eta_2 = \sqrt{\frac{\mu_r \mu_0}{\epsilon_r \epsilon_0}} = \frac{120\pi}{\sqrt{3}} \text{ hér}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 120\pi \ (\Omega) \qquad \beta_2 = \beta_0 \sqrt{\epsilon_r} = \sqrt{3}\beta_0 \text{ hér}$$

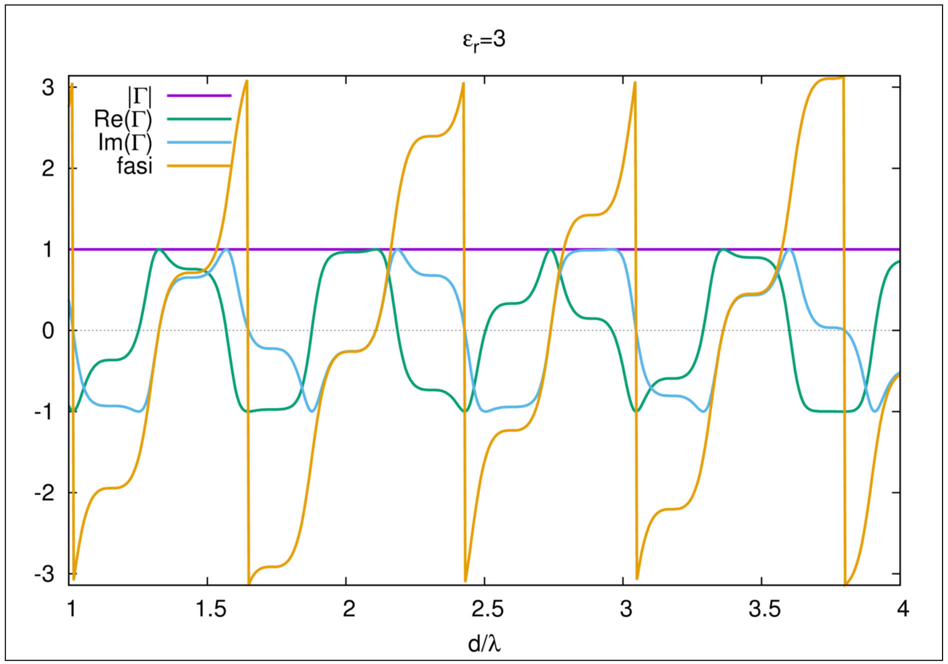
(6)

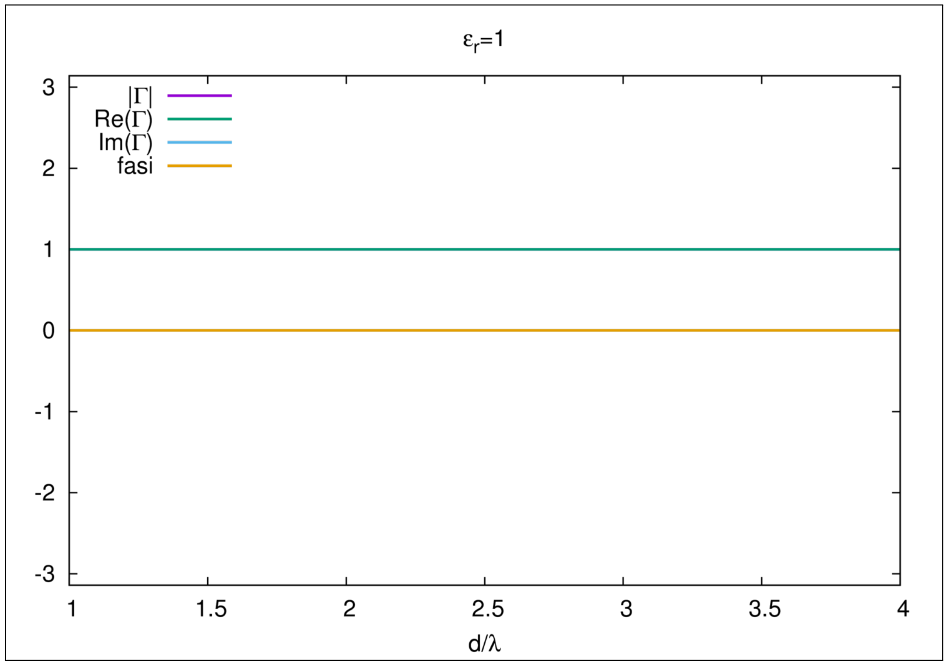
$$\beta_0 d = 2\pi \frac{d}{\lambda}$$

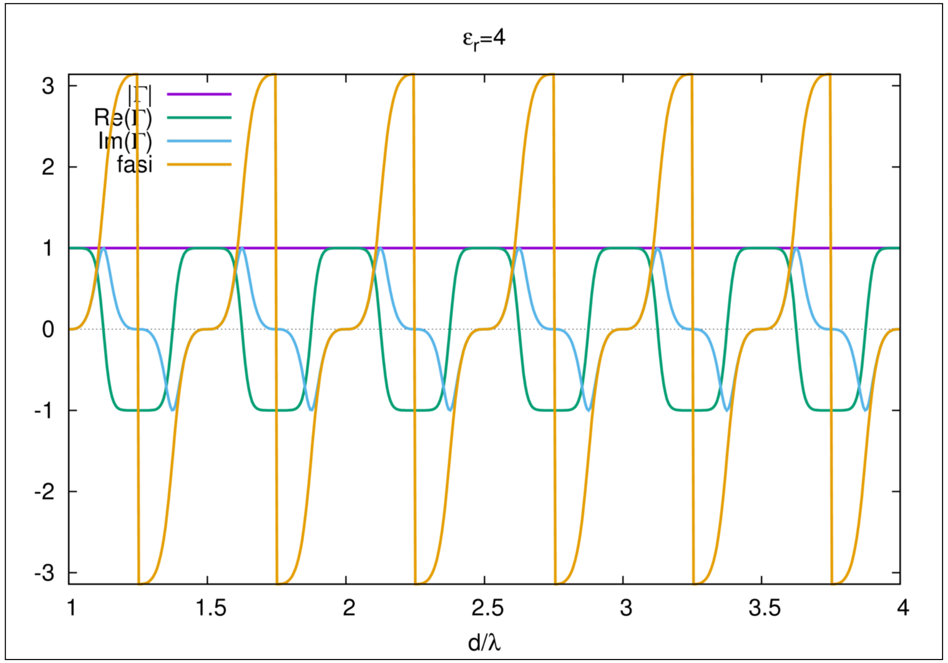
$$\beta_2 d = 2\pi \frac{d}{\lambda}\sqrt{3}, \qquad \Gamma = \frac{E_{ro}}{E_{io}}$$

Könnum líka fasahornið milli Re(Γ) og Im(Γ)

Ég geri myndir fyrir $\epsilon_r = 3.0$, 4.0, 1.0







εr=4
|Γ|
Re(Γ)
Im(Γ)
fasi
3
2
1
0
-1
-2
-3
1
1.5
2
2.5
3
3.5
4
d/λ