(1)

Þunn einangrandi kúluskel með $\rho_s(\theta) = \rho_{s0}\cos(3\theta)$

(1) Hér er heildarhleðslan

$$Q_s = \oint ds\, \rho_s(\Omega) = 2\pi\rho_{s0}\int_0^{\pi} a^2 \sin\theta \, d\theta \, \cos(3\theta)$$

$$= 2\pi a^2 \rho_{s0}\int_0^{\pi} d\theta \left\{4\sin\theta\cos^3\theta - 3\cos\theta\sin\theta\right\}$$

$= 0$ hvort heildið sem reiknað er í wxMaxima.

Yfirborðshleðslan er eins og skautuner hleðsla.

(2)

(2) Finnum V innan og utan kúlu

þurfum að leysa jöfnu Poissons

$$\nabla^2 V(R,\theta) = -\frac{1}{\epsilon_0}\rho_s(R,\theta)$$

Utan og innan kúluskeljar þurfum við því að leysa jöfnu Laplace, skeytum síðan saman lausnunum með yfirborðshleðsluna í huga. Almenna lausnir er

$$V_n(R,\theta) = \left\{A_n R^n + B_n R^{-(n+1)}\right\} P_n(\cos\theta)$$

(3)

Innan kúlu er engin punkthleðsla → $B_n = 0$ f. $\forall n$ þar

$$\rightarrow V_n^i(R,\theta) = A_n R^n P_n(\cos\theta)$$

Utan kúlu getur lausnin ekki vaxið án takmarkana

→ A_n f. $\forall n$ þar

$$V_n^o(R,\theta) = B R^{-(n+1)} P_n(\cos\theta)$$

Yfirborðshleðslan leiðir til brots í afleiðu V þ.a.

$$\hat{a}_{n2}\cdot(\bar{D}_1 - \bar{D}_2) = \rho_s$$

(4)

Notum okkur hér $\bar{D} = \epsilon_0\bar{E}$ (enginn rafsvari) og $\bar{E} = -\bar{\nabla} V$ ásamt $\hat{a}_{n2} = \hat{a}_R$ →

$$(*) \quad \left\{\partial_R V^o(R,\theta) - \partial_R V^i(R,\theta)\right\}\Big|_{R=a} = -\frac{\rho_{s0}}{\epsilon_0}\cos(3\theta)$$

þar sem við höfum summað upp í almennu lausnirnar

$$\boxed{\begin{aligned} V^o(R,\theta) &= \sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta) \qquad R > a \\ V^i(R,\theta) &= \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) \qquad R < a \end{aligned}}$$

Við þurfum að umrita $\cos(3\theta)$ sem Legendre föll.
Einfaldast er að fletta upp Chebyshev fleirliðum

$$\rightarrow T_3(\cos\theta) = \cos(3\theta) = \frac{1}{5}\left\{8P_3(\cos\theta) - 3P_1(\cos\theta)\right\}$$

Til að uppfylla jaðarskilyrðin (*) fáum við þá

$$-\sum_{n=0}^{\infty}(n+1)\frac{B_n}{a^{n+2}}P_n(\cos\theta) - \sum_{n=0}^{\infty} n A_n a^{n-1} P_n(\cos\theta)$$

Legendre fleirliðurnir eru stök í hornréttum grunni. Því einfaldast jaðarskilyrðin í

$$= -\frac{\rho_{so}}{\epsilon_0 5}\left\{8P_3(\cos\theta) - 3P_1(\cos\theta)\right\}$$

$$-2\frac{B_1}{a^3}P_1(\cos\theta) - A_1 P_1(\cos\theta) = \frac{3}{5}\frac{\rho_{so}}{\epsilon_0}P_1(\cos\theta)$$

$$-4\frac{B_3}{a^5}P_3(\cos\theta) - 3A_3 a^2 P_3(\cos\theta) = -\frac{8}{5}\frac{\rho_{so}}{\epsilon_0}P_3(\cos\theta)$$

og $B_n = 0$ og $A_n = 0$ ef $n \neq 1$ eða $n \neq 3$

$$-\frac{2}{a^3}B_1 - A_1 = \frac{3}{5}\frac{\rho_{so}}{\epsilon_0}$$

$$-\frac{4}{a^5}B_3 - 3a^2 A_3 = -\frac{8}{5}\frac{\rho_{so}}{\epsilon_0}$$

Rafstöðumættið er samfellt í $R = a$

$$V^i(a^-) = V^o(a^+)$$

$$\rightarrow A_n a^n = B_n \frac{1}{a^{n+1}} \rightarrow B_n = A_n a^{2n+1}$$

því verða jaðarskilyrðin saman

$$-2A_1 - A_1 = \frac{3}{5}\frac{\rho_{so}}{\epsilon_0} \longrightarrow A_1 = -\frac{\rho_{so}}{5\epsilon_0}$$

$$-4a^2A_3 - 3a^2A_3 = -\frac{8}{5}\frac{\rho_{so}}{\epsilon_0} \longrightarrow A_3 = \frac{8\rho_{so}}{35\epsilon_0 a^2}$$

$$\rightarrow B_1 = -\frac{\rho_{so}}{5\epsilon_0}a^3, \qquad B_3 = \frac{8\rho_{so}}{35\epsilon_0}a^5$$

$$V^o(R,\theta) = \frac{\rho_{so}}{5\epsilon_0}\left\{-\frac{a^3}{R^2}P_1(\cos\theta) + \frac{8}{7}\frac{a^5}{R^4}P_3(\cos\theta)\right\}$$

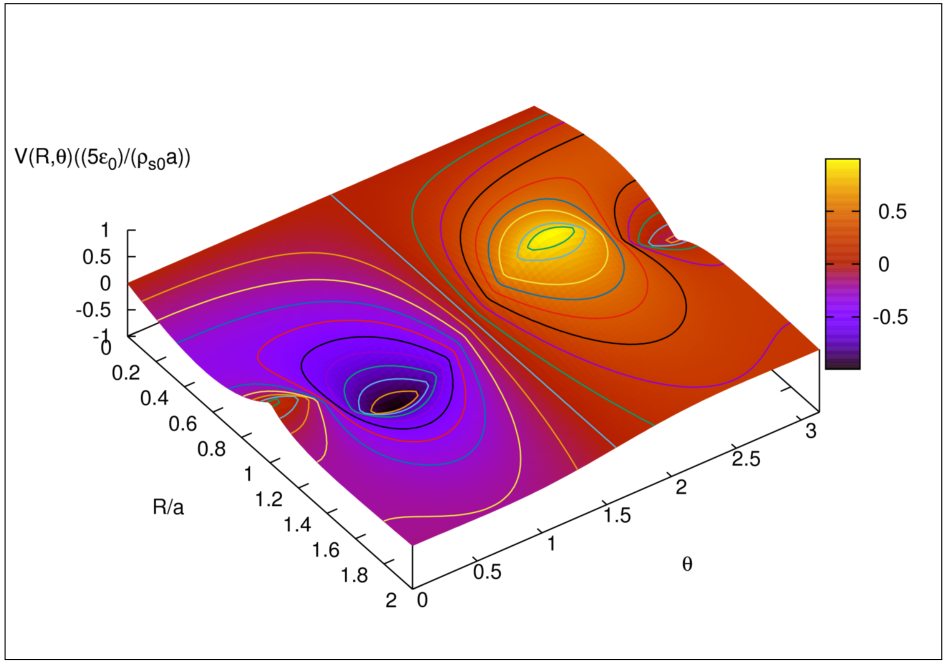
Hér má sjá bæði tvískauts og fjórskauts lið:

$$= \frac{\rho_{so}a}{5\epsilon_0}\left\{-\left(\frac{a}{R}\right)^2 P_1(\cos\theta) + \frac{8}{7}\left(\frac{a}{R}\right)^4 P_3(\cos\theta)\right\}$$

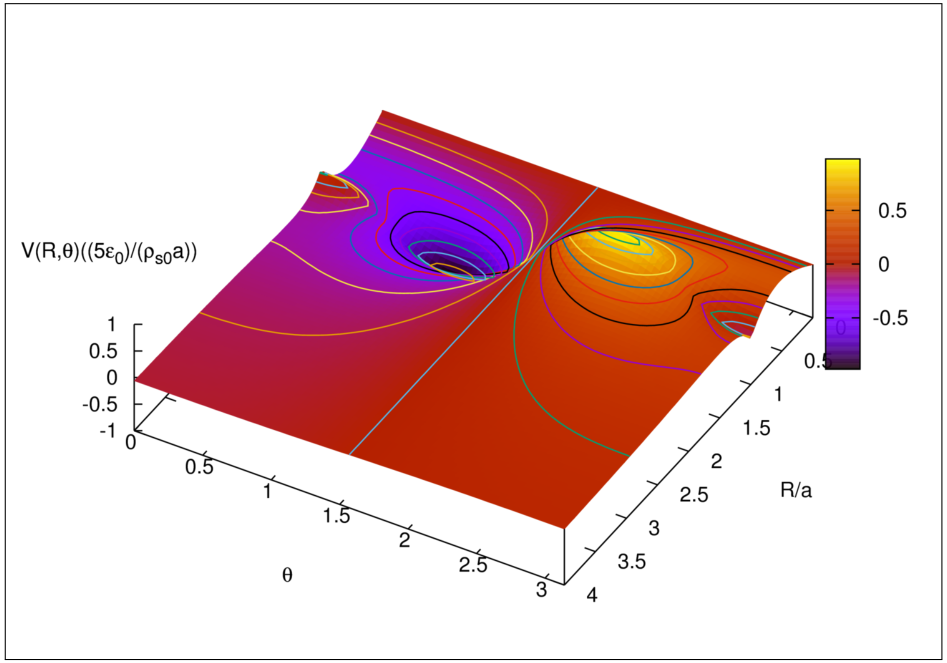
$$V^i(R,\theta) = \frac{\rho_{so}a}{5\epsilon_0}\left\{-\left(\frac{R}{a}\right)P_1(\cos\theta) + \frac{8}{7}\left(\frac{R}{a}\right)^3 P_3(\cos\theta)\right\}$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_3(\cos\theta) = \frac{1}{2}\left\{5\cos^3\theta - 3\cos\theta\right\}$$



V(R,θ)((5ε0)/(ρs0a))
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
R/a
0
0.5
1
1.5
2
2.5
3
θ
0.5
0
-0.5



V(R,θ)((5ε0)/(ρs0a))
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
2.5
3
θ
0.5
1
1.5
2
2.5
3
3.5
4
R/a
0.5
0
-0.5