① Þunn einangrandi kúluskel er með yfirborðshleðslu

$\rho_s(\theta) = \rho_{s0} \sin(3\theta)$, gefið a

finna $V(R\theta)$ innan og utan skeljar

Ég segi eina lausnar aðferð í eðlisfræði

Hér er verkefninu lýst með jöfnu Poissons

$$\nabla^2 V(R,\theta) = -\frac{1}{\epsilon_0} \rho_s(R,\theta)$$

utan og innan kúlunnar verður jafnan að jöfnu Laplace við leysum hana fyrir og stöðum síðan saman lausnunum t.þ.a. taka tillit til ρ_s á kúluyfirborðinu

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Almenna lausnin fyrir þ-einsleitt kerfi er samantekt

$$V_n(R,\theta) = \left\{ A_n R^n + B_n R^{-(n+1)} \right\} P_n(\cos\theta)$$

Innan kúlu er engin punktheild a $\rightarrow B_n = 0$ þar f. $\forall n$

$$V_n^i(R,\theta) = A_n R^n P_n(\cos\theta)$$

utan kúlu getur lausnin ekki vaxið án takmarkana $\rightarrow A_n = 0$,
þannig að við að
$V(R\to\infty) = 0$

$$V_n^o(R,\theta) = B_n R^{-(n+1)} P_n(\cos\theta)$$

Samsetning lausnanna fyrir $R=a$ verður að uppfylla

$$\hat{a}_{n2} \cdot (\bar{D}_1 - \bar{D}_2) = \rho_s$$

Við notum $\bar{E} = -\bar{\nabla} V$ og $\bar{D} = \epsilon_0 \bar{E}$ ásamt $\hat{a}_{n2} = \hat{a}_R$ ③

þá verður skilyrðin

$$\left\{ \frac{\partial}{\partial R} V^o(R,\theta) - \frac{\partial}{\partial R} V^i(R,\theta) \right\}\Big|_{R=a} = -\frac{\rho_{so}}{\epsilon_0} \sin(3\theta) \quad (*)$$

b.s.

$$V^o(R,\theta) = \sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta) \qquad R > a$$

og

$$V^i(R,\theta) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) \qquad R < a$$

Við þurfum að umskrifa $\sin(3\theta)$ yfir í P_n-röð

$$\sin(3\theta) = \sum_{n=0}^{\infty} C_n P_n(\cos\theta)$$

P_n-in eru hornrétt

$$\int_0^{\pi} \sin\theta \, d\theta \, P_n(\cos\theta) P_{n'}(\cos\theta) = \frac{\delta_{n,n'} \cdot 2}{2n+1}$$

$$\rightarrow C_\ell = \frac{2\ell+1}{2} \int_0^{\pi} \sin\theta \, d\theta \, P_\ell(\cos\theta) \sin(3\theta)$$

notum $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

$$\rightarrow C_\ell = \frac{2\ell+1}{2} \int_0^{\pi} d\theta \, P_\ell(\cos\theta) \left\{ 3\sin^2\theta + 4\sin^4\theta \right\}$$

$$C_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} dx\, P_\ell(x)\left\{3\sqrt{1-x^2} + 4\left(1-x^2\right)^{3/2}\right\}$$

á bilinu $[-1,1]$ eru $\sqrt{1-x^2}$ og $(1-x^2)^{3/2}$ jafnstæð föll

Um $P_n(x)$ gildir að $P_n(-x) = (-1)^n P_n(x)$

þá eru aðeins $C_\ell \neq 0$ fyrir $\ell = 0, 2, 4, 6, 8, \ldots$ mögulega

$P_0(\theta) = 1$

$$C_0 = \frac{1}{2}\int_0^\pi d\theta \sin\theta \cdot \sin(3\theta) = 0$$

þú er kúluskelin í heild óhlaðin og hefur aftur einskautsvogi

þetta er líka sama heildið u.þ.b og til þess að finna heildarhleðsluna

$$\int_0^\pi \sin\theta\, d\theta\, g_s(a,\theta) = 0$$

Heildið er nokkuð snúið, öll C_ℓ fyrir jöfn $\ell \geq 2$ koma í summuna. Ég reiknaði nokkur með maxima

$$C_2 = \frac{5}{2}\,\frac{3\pi}{16}$$

$$C_4 = -\frac{9}{2}\,\frac{15\pi}{256}$$

$$C_6 = -\frac{13}{2}\,\frac{21\pi}{2048}$$

$$C_8 = -\frac{17}{2}\,\frac{63\pi}{16384}$$

$$C_{10} = -\frac{21}{2}\,\frac{495\pi}{262144}$$

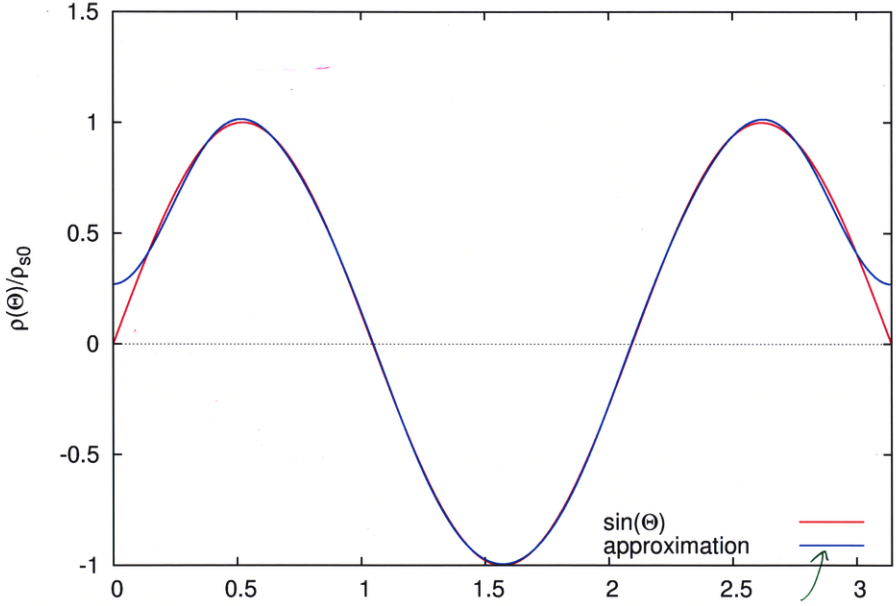
Á næstu síðu sýni ég grafík af samanburðinum fyrir

$$\rho(\theta) = \rho_{50}\sin(3\theta)$$

og

$$\sum_{n=2,4,6,8} C_n P_n(\cos\theta)\cdot \rho_{50}$$

Þar kemur í ljós að þessir liðir vega nær núna rétt þegar $\theta \to 0$ eða $\theta \to \pi$ {í kringum N og S - skaut}



Nätgun med tidrum upp i C_{10}

Við þurfum að uppfylla (*)

$$-\sum_{n=2}^{\infty}(n+1)\frac{B_n}{a^{n+2}}P_n(\cos\theta)-\sum_{n=2}^{\infty}nA_n a^{n-1}P_n(\cos\theta)=-\frac{\rho_{so}}{\epsilon_0}\sum_{n=2}^{\infty}C_nP_n(\cos\theta) \quad (**)$$

fyrir $n=\{2,4,6,8,10,\cdots\}$

þar að auki verður maður að vera samfelt í $R=a$

$$\rightarrow \sum_{n=2}^{\infty}A_n a^n P_n(\cos\theta)=\sum_{n=2}^{\infty}B_n a^{-(n+1)}P_n(\cos\theta)$$

$$\rightarrow \boxed{B_n = A_n a^{2n+1}} \quad \text{fyrir hvern lið}$$

(**) $\rightarrow$

$$\sum_{n=2}^{\infty}(2n+1)A_n a^{n-1}P_n(\cos\theta)=\frac{\rho_{so}}{\epsilon_0}\sum_{n=2}^{\infty}C_nP_n(\cos\theta)$$

Hér þarf að raða saman liðum með sama P_n

$$\rightarrow \quad (2n+1) A_n a^{n-1} = \frac{\rho_{so}}{\epsilon_0} C_n$$

$$\rightarrow \quad A_n = \frac{\rho_{so}}{\epsilon_0} \frac{C_n}{(2n+1) a^{n-1}}$$

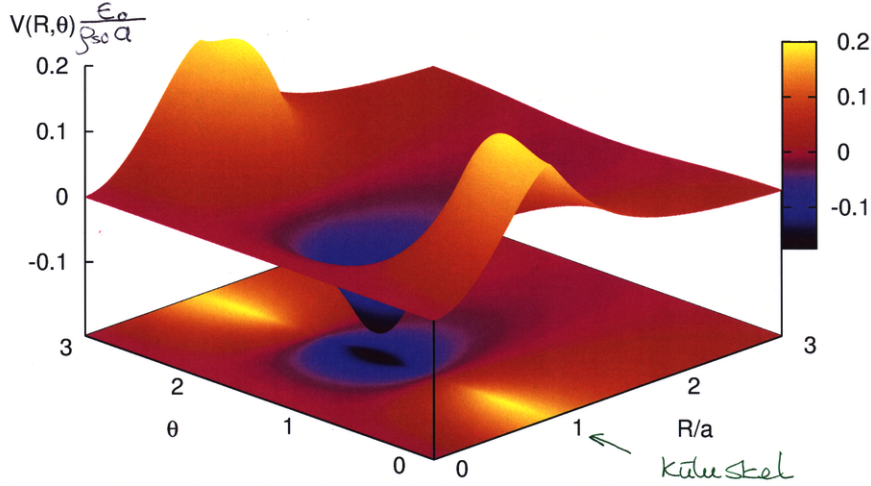
$$\rightarrow \quad B_n = \frac{\rho_{so}}{\epsilon_0} \frac{C_n}{(2n+1)} a^{n+2}$$

Því fæst

$$V^i(R,\theta) = \frac{\rho_{so}}{\epsilon_0} a \sum_{n=2}^{\infty} \frac{C_n}{2n+1} \left(\frac{R}{a}\right)^n P_n(\cos\theta)$$

$$V^o(R,\theta) = \frac{\rho_{so}}{\epsilon_0} a \sum_{n=2}^{\infty} \frac{C_n}{2n+1} \left(\frac{a}{R}\right)^{n+1} P_n(\cos\theta)$$

Nälgum med tiden upp i C_{10}



Ytri lausnin er

$$V^o(R,\theta) = \frac{\rho_{s0} a}{\epsilon_0} \sum_{n=2}^{\infty} \frac{C_n}{2n+1} \left(\frac{a}{R}\right)^{n+1} P_n(\cos\theta) \qquad R > a$$

þegar $R/a >> 1$ þá er lausnin

$$V^o(R,\theta) \longrightarrow \frac{\rho_{s0} a}{\epsilon_0} \frac{C_2}{5} \left(\frac{a}{R}\right)^3 P_2(\cos\theta)$$

matti fjörskauts, þegar nær kemur kúlunni birtast allir hærri liðir sem deyja ört út.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 13.04.2 ] */

/* [wxMaxima: input   start ] */
integrate(sin(x)*sin(3*x)*P(x), x, 0, %pi);
/* [wxMaxima: input   end   ] */

/* [wxMaxima: input   start ] */
integrate(sin(x)*sin(3*x)*((3*(cos(x))**2-1)/2), x, 0, %pi);
/* [wxMaxima: input   end   ] */

/* [wxMaxima: input   start ] */
integrate(sin(x)*sin(3*x)*((35*(cos(x))**4-30*(cos(x))**2+3)/8), x, 0, %pi);
/* [wxMaxima: input   end   ] */

/* [wxMaxima: input   start ] */
integrate(sin(x)*sin(3*x)*((231*(cos(x))**6-315*(cos(x))**4+105*(cos(x))**2-5)/16), x, 0, %pi);
/* [wxMaxima: input   end   ] */

/* [wxMaxima: input   start ] */
integrate(sin(x)*sin(3*x)*((6435*(cos(x))**8-12012*(cos(x))**6+6930*(cos(x))**4-1260*(cos(x))**2+35)/128), x, 0, %pi);
/* [wxMaxima: input   end   ] */

/* [wxMaxima: input   start ] */
integrate(sin(x)*sin(3*x)*((46189*(cos(x))**10-109395*(cos(x))**8+90090*(cos(x))**6-30030*(cos(x))**4+3465*(cos(x))**2-63)/256), x, 0, %pi);
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

Heilden til [illegible]

[illegible]

```
set term post landscape enhanced color solid 'Helvetica' 18
set output 'Pl-lidun.ps'

set key right bottom Left

set xlabel "{/Symbol Q}"
set xlabel  offset character 0, 0, 0 font "" textcolor lt -1 norotate

set ylabel "{/Symbol r}({/Symbol Q})/{/Symbol r}_{s0}"
set ylabel  offset character 0, 0, 0 font "" textcolor lt -1 rotate by -270

set xzeroaxis
set samples 5000

P2(x)  = (3*x**2-1)/2.0
P4(x)  = (35*x**4-30*x**2+3)/8.0
P6(x)  = (231*x**6-315*x**4+105*x**2-5)/16.0
P8(x)  = (6435*x**8-12012*x**6+6930*x**4-1260*x**2+35)/128.0
P10(x) = (46189*x**10-109395*x**8+90090*x**6-30030*x**4+3465*x**2-63)/256.0

c2  =  (5*3*pi)/(2*16.0)
c4  = -(9*15*pi)/(2*256.0)
c6  = -(13*21*pi)/(2*2048.0)
c8  = -(17*63*pi)/(2*16384.0)
c10 = -(21*495*pi)/(2*262144.0)

f(x) = c2*P2(x) + c4*P4(x) + c6*P6(x) + c8*P8(x) + c10*P10(x)

plot [0:pi] sin(3*x) w l lt 1 lw 2 title 'sin({/Symbol Q})',\
            f(cos(x)) w l lt 3 lw 2 title 'approximation'
#    EOF
```

Gnuplot
skrifta til að reyna
samleitni raðar fyrir
$\rho_s(a,\theta)$

```
set term post landscape enhanced color solid 'Helvetica' 18
set output 'V-Pl-lidun.ps'

unset key

set xlabel "R/a"
set xlabel  offset character 0, 0, 0 font "" textcolor lt -1 norotate

set ylabel "{/Symbol q}"
set ylabel  offset character 0, 0, 0 font "" textcolor lt -1 rotate by -270

set zlabel "V(R,{/Symbol q})"
set zlabel  offset character 4, 7, 0 font "" textcolor lt -1 norotate

set ytics border in scale 1,0.5 mirror norotate  offset character -2, 0, 0
set ytics 0,1,3
set xtics 0,1,3
set ztics  -0.2,0.1,0.2
set cbtics -0.2,0.1,0.2

set samples    200
set isosamples 200

P2(y)  = (3*y**2-1)/2.0
P4(y)  = (35*y**4-30*y**2+3)/8.0
P6(y)  = (231*y**6-315*y**4+105*y**2-5)/16.0
P8(y)  = (6435*y**8-12012*y**6+6930*y**4-1260*y**2+35)/128.0
P10(x) = (46189*x**10-109395*x**8+90090*x**6-30030*x**4+3465*x**2-63)/256.0

c2  =  (5*3*pi)/(2*16.0)
c4  = -(9*15*pi)/(2*256.0)
c6  = -(13*21*pi)/(2*2048.0)
c8  = -(17*63*pi)/(2*16384.0)
c10 = -(21*495*pi)/(2*262144.0)


Vo(x,y) = c2*P2(y)/(5.0*x**3) + c4*P4(y)/(9.0*x**5) + c6*P6(y)/(13.0*x**7) + c8*P8(y)/(17.0*x**9) + c10*P10(y)/(21.0*x**11)
Vi(x,y) = x**2*c2*P2(y)/5.0 + x**4*c4*P4(y)/9.0 + x**6*c6*P6(y)/13.0 + x**8*c8*P8(y)/17.0 + x**10*c10*P10(y)/21.0

V(x,y) = x>1 ? Vo(x,y) : Vi(x,y)

set view 57,315
set ticslevel 0.1
set pm3d at bs
splot [0:3][0:pi] V(x,cos(y)) w pm3d
#    EOF
```

Gnuplot

Skrifta fyrir $V(R,\theta)$