2.19 Líkindastraumsþéttleiki

$$J(x,t) = \frac{i\hbar}{2m}\left\{ (\partial_x \Psi^*)\Psi - \Psi^* \partial_x \Psi \right\}$$

Reikna fyrir

$$\Psi_k(x,t) = A \exp\left\{ i(kx - \omega_k t) \right\}, \quad \omega_k = \frac{\hbar k^2}{2m}$$

$$J_k(x,t) = \frac{i\hbar}{2m}|A|^2 \left\{ -ik - ik \right\} = \frac{\hbar k}{m}|A|^2$$

Í hvaða átt?

fyrir þetta einvíða verkefni er straumurinn í sömu stefnu og k, einvídd býður upp á $\pm k$

2.22 Frjáls eind, pakki

$$\Psi(x,0) = A e^{-ax^2} \qquad a \in \mathbb{R},\ a > 0$$

a) ~~Staðla~~ $\Psi(x,0)$, skilgreini $\alpha^{-2} = a$, $\alpha = \frac{1}{\sqrt{a}}$

$$|A|^2 \int_{-\infty}^{\infty} dx\, e^{-2ax^2} = |A|^2 \alpha \int_{-\infty}^{\infty} \frac{dx}{\alpha}\, e^{-2\left(\frac{x}{\alpha}\right)^2} = |A|^2 \alpha \int_{-\infty}^{\infty} du\, e^{-2u^2} = |A|^2 \alpha \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$\rightarrow |A|^2 \alpha \frac{\sqrt{\pi}}{\sqrt{2}} = 1 \quad \text{sem sé} \quad A = \sqrt{\frac{\sqrt{2}}{\alpha\sqrt{\pi}}} = \left(\frac{a\cdot 2}{\pi}\right)^{1/4}$$

b) Finna $\Psi(x,t)$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \phi(k)\, e^{i(kx - \omega_k t)}$$

(2)

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \Psi(x,0)\, e^{-ikx} = \sqrt{\frac{\sqrt{2}}{2\pi\alpha\sqrt{\pi}}} \int_{-\infty}^{\infty} dx\, e^{-\left(\frac{x}{\alpha}\right)^2 - ikx}$$

$$= \sqrt{\frac{\sqrt{2}}{2\pi\alpha\sqrt{\pi}}}\, \alpha \int_{-\infty}^{\infty} \frac{dx}{\alpha}\, \exp\left\{-\left(\frac{x}{\alpha}\right)^2 - ik\alpha\left(\frac{x}{\alpha}\right)\right\}$$

$$= \sqrt{\frac{\alpha}{\pi\sqrt{2\pi}}} \int_{-\infty}^{\infty} du\, \exp\left\{-u^2 - ik\alpha u\right\}$$

$$= \sqrt{\frac{\alpha}{\pi\sqrt{2\pi}}}\, e^{-\left(\frac{k\alpha}{2}\right)^2} \int_{-\infty}^{\infty} du\, e^{-\left(u + \frac{ik\alpha}{2}\right)^2} = \sqrt{\pi}\sqrt{\frac{\alpha}{\pi\sqrt{2\pi}}}\, e^{-\left(\frac{k\alpha}{2}\right)^2}$$

$$= \sqrt{\frac{\alpha}{\sqrt{2\pi}}}\, e^{-\left(\frac{k\alpha}{2}\right)^2} = \frac{1}{\sqrt{\sqrt{2\pi a}}}\, e^{-\left(\frac{k}{a}\right)^2}$$

Engin hliðrun pakka,
hann er jafnstæður
(samhverfur) í k
um $k=0$

$$\Phi(x,t) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\alpha}{\sqrt{2\pi}}}\int_{-\infty}^{\infty} dk\, e^{i(kx-\omega_k t)-\left(\frac{k\alpha}{2}\right)^2}$$

$$= \sqrt{\frac{\alpha}{2\pi\sqrt{2\pi}}}\int_{-\infty}^{\infty} dk\, \exp\left\{-\left(\frac{k\alpha}{2}\right)^2 - i\frac{\hbar k^2}{2m}t + i(k\alpha)\frac{x}{\alpha}\right\}$$

$$\left\{-\left(\frac{k\alpha}{2}\right)^2 - i\frac{\hbar t}{2m\alpha^2}(k\alpha)^2 + i(k\alpha)\frac{x}{\alpha}\right\} = \left\{-\left(\frac{k\alpha}{2}\right)^2\left[1 + i\frac{2\hbar t}{\alpha^2 m}\right] + i(k\alpha)\frac{x}{\alpha}\right\}$$

$$= \left\{-\left(\frac{k\alpha}{2}\right)^2\beta^2 + i(k\alpha)\frac{x}{\alpha}\right\} = -\left\{\frac{(k\alpha)\beta}{2} - \frac{i}{\beta}\left(\frac{x}{\alpha}\right)\right\}^2 - \left(\frac{x}{\beta\alpha}\right)^2$$

$$\rightarrow \Phi(x,t) = \frac{1}{\alpha}\sqrt{\frac{\alpha}{2\pi\sqrt{2\pi}}}\exp\left(-\left(\frac{x}{\beta\alpha}\right)^2\right)\int_{-\infty}^{\infty} d(k\alpha)\, \exp\left[-\left(\frac{k\alpha}{2}\beta - \frac{ix}{\beta\alpha}\right)^2\right]$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\alpha\sqrt{2\pi}}} \exp\left\{-\left(\frac{x}{\beta\alpha}\right)^2\right\} \int_{-\infty}^{\infty} du \, \exp\left[-\frac{\beta^2}{4}\left(u - \frac{ix2}{\beta^2\alpha}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi\alpha\sqrt{2\pi}}} \exp\left\{-\left(\frac{x}{\beta\alpha}\right)^2\right\} \sqrt{\pi}\, \frac{2}{\beta}$$

$$= \frac{\sqrt{2}}{\sqrt{\alpha\sqrt{2\pi}}} \exp\left\{-\left(\frac{x}{\beta\alpha}\right)^2\right\} \frac{1}{\beta} = \left(\frac{2}{\alpha^2\pi}\right)^{1/4} \exp\left\{-\left(\frac{x}{\beta\alpha}\right)^2\right\} \frac{1}{\beta}$$

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c) finna $|\Psi(x,t)|^2$ og tákna við $w^{-1} = \sqrt{\dfrac{a}{1 + \left(\frac{2\hbar a t}{m}\right)^2}}$

$$w^{-1} = \sqrt{\frac{1}{\alpha^2\left\{1 + \left(\frac{2\hbar t}{m\alpha^2}\right)^2\right\}}} \qquad \beta^2 = \left\{1 + \frac{2\hbar i t}{\alpha^2 m}\right\}$$

ég breyti þ.a. $[w] \sim L$

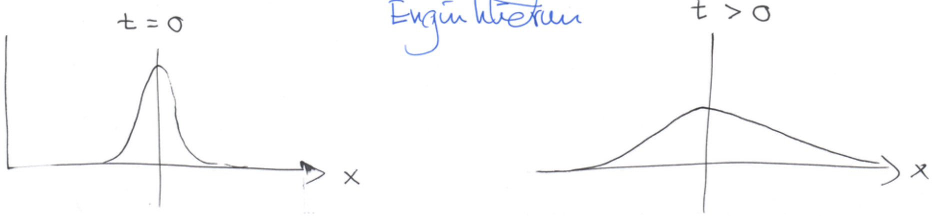
$$|\Psi(x,t)|^2 = \sqrt{\frac{2}{\alpha^2\pi}}\,\exp\left\{-\left(\frac{x}{\alpha}\right)^2\left[\frac{2}{1+\left(\frac{2\hbar t}{\alpha^2 m}\right)^2}\right]\right\}\frac{1}{\sqrt{1+\left(\frac{2\hbar t}{\alpha^2 m}\right)^2}}$$

$$= \sqrt{\frac{2}{\pi}}\,\exp\left\{-2\left(\frac{x}{w}\right)^2\right\}\frac{1}{w}, \qquad [w] = L$$

Hvernig breytist $w(t)$ með tíma

$w(0) = \alpha$ upprunaleg stærðin (náttúrulegur skali)

$w(t) \underset{t\to\infty}{\longrightarrow} \infty$



d) samhverfur pakki í $\underline{x}$ og $\underline{p}$

$\rightarrow \langle x \rangle = 0 \quad , \quad \langle p \rangle = 0$

$$\langle x^2 \rangle = \sqrt{\frac{2}{\pi w^2}} \int_{-\infty}^{\infty} dx \; x^2 \; e^{-2\left(\frac{x}{w}\right)^2}$$

$$= \sqrt{\frac{2}{\pi}} \, w^2 \int_{-\infty}^{\infty} du \, u^2 e^{-2u^2} = w^2 \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{2^{5/2}}$$

$$= w^2 \, 2^{\frac{1}{2} - \frac{5}{2}} = \frac{w^2}{4}$$

$\langle x^2 \rangle \sim w^2$

↑ sem vex með tíma

$\langle p^2 \rangle$: Frjáls eind, engin ytri kraftar breyti skriðþunga

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \phi(k)\, e^{i(kx - i\omega_k t)}$$

fast fall, samhverft, sem við fundum

Væntigildi $\langle p \rangle$ og $\langle p^2 \rangle$ eru óháð tíma

$$\Psi(x,0) = \left(\frac{2}{\alpha^2 \pi}\right)^{1/4} \exp\left\{-\left(\frac{x}{\alpha}\right)^2\right\} \qquad \text{því} \quad \beta(0) = 1$$

$$-\hbar^2 \partial_x^2 \Psi(x,0) = -\hbar^2 \left(\frac{2}{\alpha^2 \pi}\right)^{1/4} \exp\left\{-\left(\frac{x}{\alpha}\right)^2\right\} \left\{\frac{4x^2}{\alpha^4} - \frac{2}{\alpha^2}\right\}$$

$$\langle p^2 \rangle = -\sqrt{\frac{2}{\pi}}\,\frac{\hbar^2}{\alpha}\int_{-\infty}^{\infty} dx\, \exp\left\{-2\left(\frac{x}{\alpha}\right)^2\right\}\left\{\frac{4x^2}{\alpha^4} - \frac{2}{\alpha^2}\right\}$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{\hbar^2}{\alpha^2}\int_{-\infty}^{\infty} \frac{dx}{\alpha}\, \exp\left\{-2\left(\frac{x}{\alpha}\right)^2\right\}\cdot\left\{4\left(\frac{x}{\alpha}\right)^2 - 2\right\}$$

$$= \frac{\hbar^2}{\alpha^2}\left\{\frac{4}{2}\sqrt{\frac{2}{\pi}}\left[\sqrt{\frac{\pi}{2}} - \frac{\sqrt{\pi}}{2^{3/2}}\right]\right\} = \frac{\hbar^2}{\alpha^2}\left\{2\left(1-\frac{1}{2}\right)\right\} = \frac{\hbar^2}{\alpha^2}$$

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e) $$\Delta x \cdot \Delta p = \frac{w(t)}{2}\cdot\frac{\hbar}{\alpha} = \frac{\hbar}{2}\left(\frac{w(t)}{\alpha}\right)$$

$$= \frac{\hbar}{2}\sqrt{1+\left(\frac{2\hbar t}{m\alpha^2}\right)^2} \geq \frac{\hbar}{2}$$

minnst þegar $t=0$ í upphafi

og vex með tíma