(1.9)

$$\Psi(x,t) = A e^{-a\left[\frac{mx^2}{\hbar} + it\right]}$$, $A, a > 0$ Rauntölur

a) finna A

$$\int_{-\infty}^{\infty} dx\, |\Psi|^2 = A^2 \int_{-\infty}^{\infty} dx\, e^{-\frac{2amx^2}{\hbar}} = A^2 \sqrt{\frac{\hbar}{2am}} \int_{-\infty}^{\infty} d\sqrt{\frac{2am}{\hbar}}\, x\, e^{-\frac{2amx^2}{\hbar}}$$

$$= A^2 \sqrt{\frac{\hbar}{2am}} \int_{-\infty}^{\infty} du\, e^{-u^2} = A^2 \sqrt{\frac{\hbar}{2am}} \sqrt{\pi} = 1$$

$$\rightarrow A^2 = \sqrt{\frac{2am}{\hbar \pi}}$$

b) fyrir hvaða mætti V er Ψ lausn á jöfnu Schrödingers

Athugum $T\psi = \frac{p^2}{2m}\psi = -\frac{\hbar^2 d_x^2}{2m}\psi$

$$T\psi = -\frac{\hbar^2 A}{2m} d_x^2 e^{-amx^2/\hbar} = -\frac{\hbar^2 A}{2m}\left\{4\left(\frac{am}{\hbar}\right)^2 x^2 - 2\left(\frac{am}{\hbar}\right)\right\} \cdot e^{-amx^2/\hbar}$$

$$= Ae^{-\frac{amx^2}{\hbar}}\left\{-2a^2mx^2 + \hbar a\right\}$$

Schrödinger jafnan er

$H\psi = E\psi$ eða $(T+V)\psi = E\psi$

því lítur út hér að mættið sé

$$V(x) = +2ma^2x^2$$

og ortan sē $E = \hbar a$, sem einnig passar við tíma part bylgjufallsins

$$\varphi(t) = e^{-iat} = e^{-i\frac{E}{\hbar}t}$$

Passar við H.O. fyrir $a = \frac{\omega}{2}$

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c) Reikna x, x^2, p, p^2

Bylgjufallið er jafnstætt $\rightarrow \langle x \rangle = 0$, $\langle p \rangle = 0$

$$\langle p^2 \rangle = A^2 \int_{-\infty}^{\infty} dx\, e^{-\frac{2mx^2a}{\hbar}} \left[4\left(\frac{am}{\hbar}\right)^2 x^2 - 2\left(\frac{am}{\hbar}\right) \right] (-\hbar^2)$$

$$= A^2 \int_{-\infty}^{\infty} dx\, \exp\left(-\frac{2max^2}{\hbar}\right) \left[2\frac{am}{\hbar}\left(\frac{2max^2}{\hbar}\right) - 2\frac{am}{\hbar} \right] (-\hbar^2)$$

$$\langle p^2 \rangle = A^2 \sqrt{\frac{2ma}{\hbar}} \int_{-\infty}^{\infty} \sqrt{\left(\frac{2ma}{\hbar}\right)} dx \, \exp\left(-\frac{2amx^2}{\hbar}\right) \left\{ \left(\frac{2max^2}{\hbar}\right) - 1 \right\} (-\hbar^2)$$

$$= A^2 \sqrt{\frac{2ma}{\hbar}} \hbar^2 \int_{-\infty}^{\infty} du \, e^{-u^2} (1 - u^2)$$

$$= \sqrt{\frac{2am}{\hbar \pi}} \hbar^2 \sqrt{\frac{2ma}{\hbar}} \left[\frac{\sqrt{\pi}}{2} \right] = \hbar^2 \frac{am}{\hbar} = \hbar^2 \frac{m\omega}{2\hbar}$$

$$= \hbar^2 \frac{1}{2d}$$

p.s. $d = \sqrt{\frac{\hbar}{m\omega}}$ náttúrulega lengdin fyrir H.O.

$$\langle x^2\rangle = A^2 \int_{-\infty}^{\infty} dx\, x^2\, e^{-\frac{2amx^2}{\hbar}} = A^2 \sqrt{\frac{\hbar}{2ma}}\,\frac{\hbar}{2ma}\int_{-\infty}^{\infty} dx \sqrt{\frac{2am}{\hbar}} \cdot \frac{2amx^2}{\hbar}\, e^{-\frac{2amx^2}{\hbar}}$$

$$= A^2 \sqrt{\frac{\hbar}{2ma}}\,\frac{\hbar}{2ma}\int_{-\infty}^{\infty} du\, u^2 e^{-u} = A^2 \sqrt{\frac{\hbar}{2ma}}\,\frac{\hbar}{2ma}\,\frac{\sqrt{\pi}}{2}$$

$$= \frac{\hbar}{2ma}\,\frac{1}{2} = \frac{\hbar}{m\omega}\,\frac{1}{2} = \frac{d^2}{2}$$

---

d) $\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{d}{\sqrt{2}}$, $\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{\hbar}{\sqrt{2}\,d}$

$$\rightarrow \sigma_x \cdot \sigma_p = \frac{\hbar}{2}$$ minimal möglich

(1.15)

$$P(t) = \int_{-\infty}^{\infty} dx \, |\Psi(x,t)|^2 = e^{-t/\Gamma}$$

Gerum ráð fyrir $V = V_0 - i\Gamma$, $\Gamma > 0$, $\Gamma \in \mathbb{R}$

Sýna að í stað

$$d_t \int_{-\infty}^{\infty} dx \, |\Psi(x,t)|^2 = 0$$

komi nú

$$d_t P = -\frac{2\Gamma}{\hbar} P$$

Schrödinger jafnan er

$$i\hbar \partial_t \Psi = H\Psi = \left\{ \frac{p^2}{2m} + V_0 - i\Gamma \right\} \Psi$$

og

$$-i\hbar \partial_t \Psi^* = \left\{ \frac{p^2}{2m} + V_0 + i\Gamma \right\} \Psi^*$$

$$d_t |\Phi|^2 = \left(\partial_t \Phi^*\right)\Phi + \Phi^* \partial_t \Phi$$

$$= \left\{ \text{Liðirnir þegar } \frac{p^2}{2m} + V_0 \right\} - \frac{\Gamma}{\hbar}\Phi^*\Phi - \frac{\Gamma}{\hbar}\Phi^*\Phi$$

$$\rightarrow d_t P = -\frac{2\Gamma}{\hbar} P$$

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b)

$$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar} P \quad \rightarrow \quad \frac{dP}{P} = -\frac{2\Gamma}{\hbar} dt$$

með lausn $$P(t) = P_0 \exp\left\{-\frac{2\Gamma}{\hbar} t\right\}$$

Náttúrulegur tímafasti: $\frac{\hbar}{2\Gamma}$

(1.16) Sýna at

$$d_t \int_{-\infty}^{\infty} dx\, \Psi_1^* \Psi_2 = 0$$

fyrir tvær normaliserbar loysnir Schrödinger J.

$$d_t \int_{-\infty}^{\infty} dx\, \Psi_1^* \Psi_2 = \int_{-\infty}^{\infty} dx \left\{ (\partial_t \Psi_1^*) \Psi_2 + \Psi_1^* (\partial_t \Psi_2) \right\}$$

$$= \int_{-\infty}^{\infty} dx \left\{ \left(\frac{H}{-i\hbar} \Psi_1^*\right) \Psi_2 + \Psi_1^* \left(\frac{H}{i\hbar} \Psi_2\right) \right\}$$

H. er Hermitskur virki

$$= \frac{i}{\hbar} \int_{-\infty}^{\infty} dx \left\{ (H \Psi_1^*) \Psi_2 - \Psi_1^* (H \Psi_2) \right\} = 0$$