4.17

1

Í stað mættisorku

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$$

skoðum við

a)

$$V(r) = -\frac{GMm}{r}$$

M: massi sólar

m: massi jarðar

þ.a. í jöfnum fyrir H-atómið þurfum við að setja

$$GMm \leftarrow \frac{e^2}{4\pi\epsilon_0}$$

b) fyrir H-atóm er Bohr geislinn

$$a_H = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = \left(\frac{4\pi\epsilon_0}{e^2}\right)\frac{\hbar^2}{m_e}$$

fyrir J-S kerfið fæst þá

$$a_G = \frac{\hbar^2}{GMm^2} \quad , \quad G = 6.673\cdot 10^{-11}\frac{m^3}{kg\, s^2}$$

$$= \frac{(1.055\cdot 10^{-34}\, Js)^2}{6.673\cdot 10^{-11}\frac{m^3}{kg\, s^2}\cdot 2\cdot 10^{30}\,kg\,(6\cdot 10^{24}\,kg)^2}$$

$$= 2.32\cdot 10^{-138}\, m$$

c) fyrir H-atóm var

$$E_n = -R_y^H \cdot \frac{1}{n^2}$$

$$R_y^H = \left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]$$

$$\rightarrow R_y^G = \frac{m}{2\hbar^2}(GMm)^2$$

$$E_n^G = -R_y^G \frac{1}{n^2} = -\frac{m}{2\hbar^2}(GMm)^2\frac{1}{n^2}$$

Orka jarðar á hringhreyfingu

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r_0} = \text{fasti}$$

notum $F = ma$ t.þ.a. finna v

$$m\left(\frac{v^2}{r_0}\right) = \frac{GMm}{r_0^2} \rightarrow v^2 = \frac{GM}{r_0}$$

$$\rightarrow E = \frac{GMm}{2r_0} - \frac{GMm}{r_0}$$

$$= -\frac{GMm}{2r_0}$$

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setjum jafnt

$$E_n^G = E$$

$$\frac{m}{2\hbar^2}(GMm)^2\frac{1}{n^2} = \frac{GMm}{2r_0}$$

$$\rightarrow \frac{m}{2\hbar^2} GMm \frac{1}{n^2} = \frac{1}{2r_0}$$

$$\rightarrow n^2 = \frac{r_0 GMm^2}{\hbar^2} = \frac{r_0}{a_G}$$

$$\rightarrow n = \sqrt{\frac{r_0}{a_G}}$$

$$\rightarrow n \sim \sqrt{\frac{150\cdot 10^{9}\,m}{2.32\cdot 10^{-138}\,m}}$$

$$\sim 2.5\cdot 10^{74}$$

d) Jörðin geislar $n \rightarrow n-1$

hve mikil orka losnar

$$\Delta E^G = \left|E^G_{n-1} - E^G_n\right|$$

$$= +R_y^G \left\{\frac{1}{(n-1)^2} - \frac{1}{n^2}\right\}$$

$$= +R_y^G \frac{1}{n^2}\left\{\frac{n^2}{(n-1)^2} - 1\right\}$$

$$= |E_n| \cdot \left\{\frac{1}{(1-\frac{1}{n})^2} - 1\right\}$$

$$\approx |E_n| \cdot \left\{1 + \frac{2}{n} - 1\right\}$$

$$= |E_n| \cdot \frac{2}{n}$$

$$\Delta E^G = 2\cdot R_y^G \frac{1}{n^3}$$

$$R_y^G = \frac{m}{2\hbar^2}(GMm)^2 = \frac{(6\cdot 10^{24}\,kg)^3\left(6.673\cdot 10^{-11}\,\frac{m^3}{kg\,s^2}\right)^2(2\cdot 10^{30}\,kg)^2}{2\cdot(1.055\cdot 10^{-34}\,js)^2}$$

$$\approx 1{,}73\cdot 10^{182}\,J$$

$$\to \Delta E^G \approx 2\cdot R_y^G\,\frac{1}{n^3} \sim 2.1\cdot 10^{-41}\,J.$$

Hvaða bylgjulengd myndi þetta samsvara fyrir þyngdarsvið?

$$\Delta E^G = \hbar\omega = h\frac{\omega}{2\pi} = h\nu$$

$$= h\frac{c}{\lambda}$$

$$\to \lambda = \frac{hc}{\Delta E^G} = \frac{(1.055\cdot 10^{-34}\,js\cdot 2\pi)\,3\cdot 10^8\,m}{2.1\cdot 10^{-41}\,J} \approx 9.48\cdot 10^{15}\,m$$

$$\simeq 1 \text{ Ljósár}$$

4.55 (5)

Rafeindi vetni í

$$R_{21}\left\{\sqrt{\tfrac{1}{3}}\, Y_{10}\,\chi_{+} + \sqrt{\tfrac{2}{3}}\, Y_{11}\,\chi_{-}\right\}$$

a) Mæliniðurstöður L^2?

Báðir hlutir ástandsins eru eiginástönd L^2 með eigingildi $\hbar^2 1(1+1) = 2\hbar^2$, líkindin á þeirri mæliniðurstöðu eru því 1.

b) Mæliniðurstöður L_z?

Báðir hlutir ástandsins eru eiginástönd L_z, en með mismunandi eigingildi. Við fáum því

$\hbar \cdot 0$ með líkindunum $\frac{1}{3}$

$\hbar$ með líkindunum $\frac{2}{3}$

heildarlíkurnar eru því 1

c) S^2 ?

Báðir spuna pottar ástandsins eru eiginástönd S^2 með sama eigingildi → mæling gefur

$$\hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3\hbar^2}{4} \quad \text{með líkum } 1$$

d) S_z

Báðir pottar eru eiginástönd S_z, en með mismunandi eigingildum, því ~~fæst~~

$$+\frac{\hbar}{2} \quad \text{með líkum } \frac{1}{3}$$

$$-\frac{\hbar}{2} \quad \text{með líkum } \frac{2}{3}$$

e) J^2

Hér vandast málið, ástöndin eru ekki eigin ástönd J^2.
Við þurfum því að liða ástandið í þeim

$Y_{10}\chi_+$ stendur fyrir $|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle$

$Y_{11}\chi_-$ — " — $|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle$

táknum $|l,m\rangle|s,m_s\rangle$

Í báðum tilfellum fæst að $m+m_s=\frac{1}{2}$, því í samanburði við jöfnu (4.184) fæst aðeins að $j=\frac{3}{2}$ eða $\frac{1}{2}$

Við þurfum því að nota (4.186) t.þ.a finna

$$|l,m\rangle|s,m_s\rangle = \sum_j C^{l,s,j}_{m,m_s,m_j}\,|j,m_j\rangle$$

$$|1,0\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \sum_j C^{1,\frac{1}{2},j}_{0,\frac{1}{2},m_j}\,|j,m_j\rangle$$

samkvæmt töflu 4.8

$$= \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$|1,1\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \sum_j C^{1,\frac{1}{2},j}_{1,-\frac{1}{2},m_j}\,|j,m_j\rangle$$

$$= \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

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og heildar hverfiþunga hluti ástandsins er því í $\{|j,m_j\rangle\}$-grunni

$$\left[\sqrt{\tfrac{1}{3}}\,|1,0\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\,|1,1\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\right]$$

$$= \sqrt{\tfrac{1}{3}}\left\{\sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right\} + \sqrt{\tfrac{2}{3}}\left\{\sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right\}$$

$$= \frac{2\sqrt{2}}{3}\left|\overset{j}{\tfrac{3}{2}},\overset{m_j}{\tfrac{1}{2}}\right\rangle + \frac{1}{3}\left|\overset{j}{\tfrac{1}{2}},\overset{m_j}{\tfrac{1}{2}}\right\rangle$$

Ástandið er því liðað í eiginástöndum J^2 og J_z og við sjáum að mæling getur

gildi $\frac{3}{2}\left(\frac{5}{2}\right)\hbar^2$ með líkum $\frac{4\cdot 2}{9} = \frac{8}{9}$

$\frac{1}{2}\left(\frac{3}{2}\right)\hbar^2$ með líkum $\frac{1}{9}$

f) mæling á J_z getur

$\frac{\hbar}{2}$ með líkum 1

bæðir þættirnir eru eiginástönd J_z með sama eigin-gildi

g) Ástandið var $R_{21}\left\{\sqrt{\frac{1}{3}}\,Y_{10}\chi_+ + \sqrt{\frac{2}{3}}\,Y_{11}\chi_-\right\}$

líkindaþéttleikinn

fyrir því að finna eindina er

$$|R_{21}|^2\left\{\frac{1}{3}|Y_{10}(\Omega)|^2 + \frac{2}{3}|Y_{11}(\Omega)|^2\right\}$$

þar sem $\chi_+^*\chi_- = 0$ og $\chi_-^*\chi_+ = 0$

$\chi_+\chi_+ = 1$ og $\chi_-\chi_- = 1$

$$\frac{1}{24}\frac{1}{a^3}\left(\frac{r}{a}\right)^2 e^{-\frac{r}{a}}\left[\frac{1}{4\pi}\cos^2\theta + \frac{2}{8\pi}\sin^2\theta\right]$$

$$= \frac{1}{96\pi}\frac{1}{a^3}\left(\frac{r}{a}\right)^2 e^{-\frac{r}{a}}$$

h) líkurnar á að mæla fjarlægð r og S_z með $+\frac{\hbar}{2}$

$$|R_{21}|^2 \frac{1}{3} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \, |Y_{10}|^2$$

$$= \frac{1}{24} \frac{1}{a^3} \left(\frac{r}{a}\right)^2 e^{-\frac{r}{a}} \frac{2\pi}{3} \int_0^{\pi} d\theta \sin\theta \cos^2\theta \frac{3}{4\pi}$$

$$= \frac{1}{48} \frac{1}{a^3} \left(\frac{r}{a}\right)^2 e^{-\frac{r}{a}} \boxed{\int_0^{\pi} d\theta \sin\theta \cos^2\theta}$$

$$= \frac{1}{48} \frac{1}{a^3} \left(\frac{r}{a}\right)^2 e^{-\frac{r}{a}} \frac{2}{3} = \frac{1}{72a^3} \left(\frac{r}{a}\right)^2 e^{-\frac{r}{a}}$$