4.22

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a) Hvað er $L_+ Y_{\ell\ell}$?

ℓ er hæsta m-gildið

$\rightarrow L_+ Y_{\ell\ell} = 0$

b) Nota $L_+ Y_{\ell\ell} = 0$

með

$$L_\pm = \pm \hbar e^{\pm i\phi}\left\{\frac{\partial}{\partial\theta} \pm i \cot\theta \frac{\partial}{\partial\phi}\right\}$$

og

$$L_z Y_{\ell\ell} = \hbar \ell Y_{\ell\ell}$$

til að ákvarða $Y_{\ell\ell}$

(1)

$$\hbar e^{i\phi}\left\{\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi}\right\} Y_{\ell\ell}(\Omega) = 0$$

(2)

$$-i\hbar \frac{\partial}{\partial\phi} Y_{\ell\ell}(\Omega) = \hbar \ell Y_{\ell\ell}(\Omega)$$

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(2) $\rightarrow i\partial_\phi Y_{\ell\ell}(\Omega) = -\ell Y_{\ell\ell}(\Omega)$

notum í (1)

$$\hbar e^{i\phi}\left\{\partial_\theta - (\cot\theta)\ell\right\} Y_{\ell\ell}(\Omega) = 0$$

$$\rightarrow \left\{\partial_\theta - \ell \cot\theta\right\} Y_{\ell\ell} = 0$$

En gleymum ekki að (2) gefur

$$i\partial_\phi Y_{ll}(\Omega) + l\,Y_{ll}(\Omega) = 0$$

$$\rightarrow Y_{ll}(\Omega) = f(\theta)\,e^{il\phi}$$

því er seinni jafnan

$$\left\{d_\theta - l\cot\theta\right\} f(\theta) = 0$$

$$\frac{df}{f} = l\cot\theta\, d\theta$$

með lausn

$$\ln(f) = l\ln(\sin\theta)$$

$$\rightarrow f(\theta) = \sin^l\theta$$

heildarlausnin er því

$$Y_{ll}(\Omega) = \sin^l\theta\, e^{il\phi} A$$

þarsem A er stöðlunarstuðull.

c) finna A

$$\int d\Omega\, |Y_{ll}(\Omega)|^2 = 1$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \sin^{2l}\theta\, A^2$$

$$= A^2 2\pi \int_0^{\pi} d\theta\, \sin^{2l+1}\theta$$

$$= A^2 4\pi \int_0^{\pi/2} d\theta \sin^{2l+1}\theta \overset{\uparrow}{=} A^2 4\pi \frac{(2l)!!}{(2l+1)!!} = 1$$

(GR 3.621.4)

$$\rightarrow A^2 = \frac{(2l+1)!!}{(2l)!!\ 4\pi} \rightarrow A = \sqrt{\frac{(2l+1)!!}{4\pi\ (2l)!!}}$$

$P_l^l(\cos\theta) = (2l-1)!! \sin^l\theta$ þ.v. sést frá (4.32)

$$Y_{ll}(\Omega) = \epsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{1}{(2l)!}}\ e^{il\theta} (2l-1)!! \sin^l\theta$$

$$= \epsilon \sqrt{\frac{(2l+1)(2l-1)!!\ (2l-1)!!}{4\pi\ (2l)!}}\ e^{il\theta} \sin^l\theta$$

$$= \epsilon \sqrt{\frac{(2l+1)!!}{4\pi\ (2l)!!}}\ e^{il\theta} \sin^l\theta$$

Ég hendi í burtu 2 heildunarstuðlum án skýringa

Með þeim verða eigin föllin ekki korrett

H-atóm

Rafeind er í ástandi

$$|\mu\rangle = \left\{4|100\rangle + 3|211\rangle - |210\rangle + \sqrt{10}\,|21\,{-1}\rangle\right\}\frac{1}{6}$$

þar sem $|n\ell m\rangle$ eru eigin ástönd H-atóms (rafeinder í H-...)

$$\langle\mu|\mu\rangle = \left\{16 + 9 + 1 + 10\right\}\frac{1}{36} = 1$$

Svo ástandið er staðlað

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a) Finna væntigildi orku rafeindarinnar

$$H|n\ell m\rangle = E_n|n\ell m\rangle \quad \text{þ.a.} \quad E_n = -R_y\frac{1}{n^2}$$

$$\langle\mu|H|\mu\rangle = \frac{1}{36}\left\{16E_1 + 9E_2 + E_2 + 10E_2\right\} = \frac{1}{36}\left\{16E_1 + 20E_2\right\}$$

$$\rightarrow \langle \mu | H | \mu \rangle = -\frac{Ry}{36}\left\{16 \cdot \frac{1}{1} + 20 \cdot \frac{1}{4}\right\} = -\frac{Ry}{36}\left\{21\right\}$$

$$= -Ry \cdot \frac{21}{36} = -Ry \cdot \frac{7}{12}$$

b) Väntigärdi L^2 ?

$|nlm\rangle$ eru eiginástönd L^2 með eigingildi $\hbar^2 l(l+1)$

$$L^2 \cdot |nlm\rangle = \hbar^2 l(l+1)|nlm\rangle$$

$$\langle \mu | L^2 | \mu \rangle = \frac{\hbar^2}{36}\left[16 \cdot 0 + 20 \cdot 1(1+1)\right]$$

$$= \frac{\hbar^2}{36}\left\{40\right\} = \hbar^2 \cdot \frac{10}{9}$$

c) Ventigi2di L_z

$$L_z |nlm\rangle = \hbar m |nlm\rangle$$

$$\Rightarrow \langle \mu | L_z | \mu \rangle = \frac{\hbar}{36} \left\{ 16 \cdot 0 + 9 \cdot 1 + 1 \cdot 0 - 10 \cdot 1 \right\}$$

$$= \frac{\hbar}{36} (-1) = -\hbar \frac{1}{36}$$