2.12 Finna $\langle p \rangle, \langle p^2 \rangle, \langle x \rangle, \langle x^2 \rangle$ og $\langle T \rangle$
fyrir eiginástönd H.O.
{nota virkjana a^+ og a^-}

náttúrulegi lengdarstallinn
$$a = \sqrt{\frac{\hbar}{m\omega}}$$

$$x = \frac{a}{\sqrt{2}}(a_+ + a_-)$$

$$p = \frac{i\hbar}{\sqrt{2}\,a}(a_+ - a_-)$$

$$\langle x \rangle = \frac{a}{\sqrt{2}}\int_{-\infty}^{\infty} dx\, \psi_n^* (a_+ + a_-)\psi_n = \frac{a}{\sqrt{2}}\int_{-\infty}^{\infty} dx\, \psi_n^* \left\{\sqrt{n+1}\,\psi_{n+1} + \sqrt{n}\,\psi_{n-1}\right\}$$

$$= 0$$

því eigin ástöndin eru hornrétt

$$\langle p \rangle = \frac{\hbar}{\sqrt{2}\,a} \int dx\, \psi_n^* \,(a_+ - a_-)\,\psi_n = 0$$

$$x^2 = \frac{a^2}{2}\left[a_+^2 + a_+ a_- + a_- a_+ + a_-^2 \right]$$

hinir liðirnir hverfa

$$\langle x^2 \rangle = \frac{a^2}{2} \int_{-\infty}^{\infty} dx\, \psi_n^* \left[a_+ a_- + a_- a_+ \right] \psi_n$$

$$= \frac{a^2}{2} \left[\sqrt{n}\sqrt{n} + \sqrt{n+1}\sqrt{n+1} \right] = \frac{a^2}{2}\left\{ n + n + 1 \right\}$$

$$= a^2 \left(n + \tfrac{1}{2} \right)$$

$$\langle p^2 \rangle = \frac{\hbar^2}{2a^2} \int dx\, \psi_n^* \left\{ + a_+ a_- + a_- a_+ \right\} \psi$$

$$= + \frac{\hbar^2}{2a^2} \left\{ \sqrt{n}\sqrt{n} + \sqrt{n+1}\sqrt{n+1} \right\} = \frac{\hbar^2}{a^2} \left(n + \tfrac{1}{2} \right)$$

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{\hbar^2}{2ma^2}\left(n+\frac{1}{2}\right) = \frac{\hbar^2 m\omega}{2m\hbar}\left(n+\frac{1}{2}\right) = \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right)$$

$\langle T \rangle + \langle V \rangle = \langle H \rangle = \hbar\omega\left(n+\frac{1}{2}\right)$ eins og búast mátti við

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a \cdot \sqrt{n+\frac{1}{2}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar}{a}\sqrt{n+\frac{1}{2}}$$

$\rightarrow \quad \sigma_x \cdot \sigma_p = \hbar\left(n+\frac{1}{2}\right) \geq \frac{\hbar}{2}$ fyrir öll $n = 0, 1, 2, \dots$

2.21

$$\Psi(x,0) = A e^{-\alpha|x|}, \quad A \text{ og } \alpha \text{ eru jákvæðar rauntölur}$$

Normunin í skiladæmi 1.5, $A = \sqrt{\alpha}$

b) Frjálsri eind er lýst með $\Psi(x,0)$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \phi(k) e^{i(kx-\omega(k)t)}$$

þ.s. $\omega(k) = \frac{\hbar k^2}{2m}$, Finna $\phi(k)$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \Psi(x,0) e^{-ikx}$$

$$\phi(k) = \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\alpha|x| - ikx} = \sqrt{\frac{\alpha}{2\pi}} \left\{ \int_{-\infty}^{0} dx\, e^{\alpha x - ikx} + \int_{0}^{\infty} dx\, e^{-\alpha x - ikx} \right\}$$

$$\text{poi } |x| = \begin{cases} -x & \text{p. } x < 0 \\ x & \text{p. } x > 0 \end{cases}$$

$$\phi(k) = \sqrt{\frac{\alpha}{2\pi}} \left\{ \frac{e^{\alpha x - ikx}}{\alpha - ik} \Bigg|_{-\infty}^{0} + \frac{e^{-\alpha x - ikx}}{-\alpha - ik} \Bigg|_{0}^{\infty} \right\}$$

$$= \sqrt{\frac{\alpha}{2\pi}} \left\{ \frac{1}{\alpha - ik} + \frac{-1}{-\alpha - ik} \right\} = \sqrt{\frac{\alpha}{2\pi}} \left\{ \frac{1}{\alpha - ik} + \frac{1}{\alpha + ik} \right\}$$

$$= \sqrt{\frac{\alpha}{2\pi}} \left\{ \frac{2\alpha}{\alpha^2 + k^2} \right\}$$

því fæst

$$\Psi(x,t) = \sqrt{\frac{\alpha}{2\pi}} \, \frac{2\alpha}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{e^{i(kx - \omega(k)t)}}{\alpha^2 + k^2}$$

Markgildi α stórt eða lítið

$$\Psi(x,t) = \frac{\sqrt{\alpha}}{\sqrt{\pi}} \, \frac{\alpha^2}{\alpha^2} \int_{-\infty}^{\infty} \frac{dk}{\alpha} \, \frac{e^{i(\frac{k}{\alpha}(\alpha x) - \omega(k)t)}}{1 + (\frac{k}{\alpha})^2}$$

(*)

$$= \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} du \, \frac{e^{i(u \cdot (\alpha x) - \omega(u)\alpha^2 t)}}{1 + u^2}$$

Stórt α -> þröngt $\Phi(x,0)$

(*) -> heildað yfir flatarbylgjur á breiðu k-bili

-> mikil tvístrun

Lítið α -> breitt $\Phi(x,0)$

(*) -> heildað yfir flatarbylgjur á þröngu k-bili

-> lítil tvístrun