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Eind í óendanlegum brunni

$$\Psi(x,0) = A\left\{\psi_1(x) + \psi_2(x)\right\}$$

a) Staðla $\Psi(x,0)$

Rauntölu föll

$$\int_0^a dx\, |\Psi(x,0)|^2 = |A|^2 \int_0^a dx \left\{ |\psi_1(x)|^2 + |\psi_2(x)|^2 + \underbrace{2\psi_1(x)\psi_2(x)}_{=0} \right\}$$

$=0$ (heildið) því eru hornrétt

eru stöðluð

$$= |A|^2 \left\{ 1 + 1 \right\} = 1$$

$$\rightarrow |A|^2 = \frac{1}{2} \quad \text{og} \quad A = \frac{1}{\sqrt{2}}$$

b) finna $\Psi(x,t)$ og $|\Psi|^2$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} = n^2\hbar\omega \quad \text{ef} \quad \omega = \frac{\pi^2\hbar}{2ma^2}$$

$$E_1 = \hbar\omega_1 = \hbar\omega, \quad \omega_1 = \omega$$

$$E_2 = \hbar\omega_2 = 4\hbar\omega, \quad \omega_2 = 4\omega$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-i\omega_1 t} + \psi_2(x)e^{-i\omega_2 t}\right]$$

$$|\Psi(x,t)|^2 = \frac{1}{2}\left[\psi_1^2(x) + \psi_2^2(x) + \psi_1(x)\psi_2(x)\left(e^{it(\omega_1-\omega_2)} + e^{-it(\omega_1-\omega_2)}\right)\right]$$

$$= \frac{1}{2}\left[\psi_1^2(x) + \psi_2^2(x) + 2\psi_1(x)\psi_2(x)\cos\left((\omega_1-\omega_2)t\right)\right]$$

$$|\Psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\pi\frac{x}{a}\right) + \sin^2\left(2\pi\frac{x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right]$$

c) Reikna $\langle x \rangle$

$$\langle x \rangle = \int_0^a dx\ \Psi^*(x) \times \Psi(x)$$

$$= \frac{1}{2}\int_0^a dx \left[\psi_1 e^{+i\omega_1 t} + \psi_2 e^{+i\omega_2 t}\right] \times \left[\psi_1 e^{-i\omega_1 t} + \psi_2 e^{-i\omega_2 t}\right]$$

$$= \int_0^a dx \times |\Psi(x,t)|^2$$

$$= \frac{1}{a}\int_0^a dx \times \left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right]$$

$$= a\int_0^1 du\, u\left\{\sin^2(\pi u) + \sin^2(2\pi u) + 2\sin(\pi u)\sin(2\pi u)\cos(3\omega t)\right\}$$

Notam

$$\int_0^1 du\, u \sin^2(\pi u) = \frac{1}{8\pi^2}\left\{1 - 1 + 2\pi^2\right\} = \frac{1}{4}$$

$$\int_0^1 du\, u \sin^2(2\pi u) = \frac{1}{4}$$

$$\int_0^1 du\, u \sin(\pi u)\sin(2\pi u) = -\frac{-1+9}{18\pi^2} - \frac{4}{9\pi^2}$$

$$= \frac{-8-8}{18\pi^2} = -\frac{16}{18\pi^2}$$

$$= -\frac{8}{9\pi^2}$$

$$\langle x \rangle = a\left\{\frac{1}{2} - \frac{16}{9\pi^2}\cos(3\omega t)\right\}$$

$$= \frac{a}{2}\left\{1 - \frac{32}{9\pi^2}\cos(3\omega t)\right\}$$

$$\simeq \frac{a}{2}\left\{1 - 0{,}36 \cdot \cos(3\omega t)\right\}$$

$$\max\left\{\langle x \rangle\right\} \simeq a \cdot 0{,}68$$

$$\min\left\{\langle x \rangle\right\} \simeq a \cdot 0.32$$

utslaget er $\quad \frac{a}{2}\frac{32}{9\pi^2} \simeq 0{,}18\, a$

d) Reikna $\langle p \rangle$

$$\langle p \rangle = \int_0^a dx\, \Psi^*(x,t) \left\{ -i\hbar \partial_x \Psi(x,t) \right\}$$

skoðum aðeins

$$\partial_x \Psi(x,t) = \frac{1}{\sqrt{2}} \left\{ \partial_x \psi_1 e^{-i\omega_1 t} + \partial_x \psi_2 e^{-i\omega_2 t} \right\}$$

$$\partial_x \psi_1 \sim \cos\left(\frac{\pi x}{a}\right) \cdot \frac{\pi}{a}$$

$$\partial_x \psi_1 \sim \cos\left(\frac{2\pi x}{a}\right) \cdot \frac{2\pi}{a}$$

$$\langle p \rangle = -i\hbar \frac{1}{a} \int_0^a dx \left\{ \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right) \cdot \frac{\pi}{a} + \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right) \cdot \frac{2\pi}{a} \right.$$

$$\left. + \frac{2\pi}{a} \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right) e^{it(\omega_1-\omega_2)} + \frac{\pi}{a} \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right) e^{-it(\omega_1-\omega_2)} \right\}$$

Notum

$$\int_0^1 du \, \mathrm{Sin}(\pi u) \, \mathrm{Cos}(\pi u) = 0$$

$$\int_0^1 du \, \mathrm{Sin}(2\pi u) \, \mathrm{Cos}(2\pi u) = 0$$

$$\int_0^1 du \, \mathrm{Sin}(\pi u) \, \mathrm{Cos}(2\pi u) = -\frac{1}{3\pi} - \frac{-1+3}{6\pi} = -\frac{2}{3\pi}$$

$$\int_0^1 du \, \mathrm{Sin}(2\pi u) \, \mathrm{Cos}(\pi u) = \frac{2}{3\pi} - \frac{-4}{6\pi} = \frac{4}{3\pi}$$

$$\langle p \rangle = -i\hbar \frac{\pi}{a}\frac{4}{3\pi}\left\{-2i \,\mathrm{Sin}((\omega_1-\omega_2)t)\right\}$$

$$= \frac{\hbar}{a}\frac{8}{3}\,\mathrm{Sin}(3\omega t) \leftarrow \text{Rauntala}$$

e) Moli ortu til helminga fo ég gildin $E_1 = \hbar\omega$ og $E_2 = 4\hbar\omega$

ψ_1 og ψ_2 hafa sama vogið í $\Psi(x,0)$

fina $\langle H \rangle$

$$\langle H \rangle = \int dx\, \Psi^*(x,t)\, H\, \Psi(x,t) = \frac{1}{2}\left\{E_1 + E_2\right\}$$

$$= \frac{5\hbar\omega}{2}$$

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Eind í óendanlegum brunni með upphafsb.

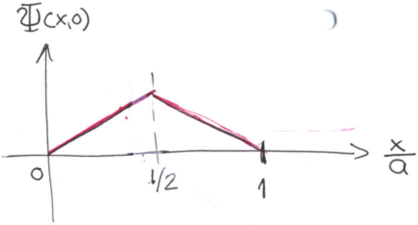
$$\Psi(x,0) = \begin{cases} Ax & 0 \leq x \leq \frac{a}{2} \\ A(a-x), & \frac{a}{2} \leq x \leq a \end{cases}$$

a) Teikna $\Psi(x,0)$ og reikna A

$$\int_0^a dx\, |\Psi(x,0)|^2 = A^2 \int_0^{a/2} dx\, x^2 + A^2 \int_{a/2}^{a} dx\, (a-x)^2$$

$$= A^2 \left\{ a^3 \int_0^{1/2} du\, u^2 + a^3 \int_{1/2}^{1} du \cdot (1-u)^2 \right\}$$

$$= A^2 a^3 \left\{ \frac{1}{24} + \frac{1}{24} \right\} = A^2 a^3 \frac{1}{12} = 1 \rightarrow A = \sqrt{\frac{12}{a^3}}$$



b) fórma $\Psi(x,t)$

$$\Psi(x,0) = \sum_{n=1}^{\infty} C_n \psi_n(x)$$

$$C_n = \int_0^a dx\, \psi_n^*(x)\, \Psi(x,0)$$

$$C_n = \int_0^{a/2} dx\, \psi_n(x)\, A x + \int_{a/2}^{a} dx\, \psi_n(x)\, A(a-x)$$

$$= \sqrt{\frac{2}{a}}\sqrt{\frac{12}{a^3}} \int_0^{a/2} dx\, \sin\left(\frac{n\pi x}{a}\right) x$$

$$+ \sqrt{\frac{2}{a}}\sqrt{\frac{12}{a^3}} \int_{a/2}^{a} dx\, \sin\left(\frac{n\pi x}{a}\right)(a-x)$$

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$$C_n = \sqrt{\frac{2}{a}}\sqrt{\frac{12a^2}{a^3}}\left[\int_0^{1/2} du \,\mathrm{Sin}(n\pi u)\cdot u + \int_{1/2}^{1} du \,\mathrm{Sin}(n\pi u)\cdot(1-u)\right]$$

heildi fyrir Fourier stuðla, þegar þau eru leyst fast

$$\Psi(x,t) = \sum_{n=1}^{\infty} C_n \psi_n(x) e^{-in^2\omega t} \quad \text{ef} \quad \omega = \frac{\pi^2\hbar}{2ma^2}$$

og $\psi_n(x) = \sqrt{\frac{2}{a}}\,\mathrm{Sin}\left(\frac{n\pi}{a}\right)$

$C_n = \sqrt{24}\, f(n)$

fall af n sem við byrjum að finna t.d. (GR 2.634.1) eða

$C_n = 0$ fyrir $n = 2, 4, 6, 8$ vegna samhverfu, og einfaldlega

$$\int_0^{1/2} du\, \sin(n\pi u)\cdot u = \mathrm{Im}\left\{ \int_0^{1/2} du\, e^{in\pi u}\cdot u \right\} = I_1$$

$$\int_{1/2}^{1} du\, \sin(n\pi u)(1-u) = \mathrm{Im}\left\{ \int_{1/2}^{1} du\, e^{in\pi u}(1-u) \right\} = I_2$$

$$I_1 = \mathrm{Im}\left\{ -\frac{1}{n^2\pi^2}\left(\frac{(i\pi n-2)}{2}(i)^n - 1 \right) \right\} = \frac{1}{n^2\pi^2}\mathrm{Im}(i)^n$$

$$I_2 = \mathrm{Im}\left\{ -\frac{1}{n^2\pi^2}\left((-1)^n - \frac{(i\pi n+2)}{2}(i)^2 \right) \right\} = \frac{1}{n^2\pi^2}\mathrm{Im}(i)^n$$

$$I_1 + I_2 = \frac{2}{n^2\pi^2}\mathrm{Im}(i)^n = \frac{2}{n^2\pi^2}(-1)^p \quad \text{if} \quad n = 2p+1$$

$$C_n = \sqrt{\frac{2}{a}}\,\frac{\sqrt{12}\,a^2}{\sqrt{a^3}}\;\frac{2}{n^2\pi^2}(-1)^p \quad \text{ef} \quad n = 2p+1$$

$$= \sqrt{24}\,\frac{2}{n^2\pi^2}(-1)^p = \sqrt{96}\,\frac{(-1)^p}{n^2\pi^2} \qquad p = 0, 1, 2, \ldots$$

$$\rightarrow \boxed{\Psi(x,t) = \sum_{p=0}^{\infty} \sqrt{96}\,\frac{(-1)^p}{(2p+1)^2\pi^2}\sqrt{\frac{2}{a}}\,\sin\left(\frac{(2p+1)\pi}{a}\right) e^{-i(2p+1)^2\omega t}}$$

c) Líkindi þess að mæla E_1 er $|$liðunarstuðullinn$|^2$ við Ψ_1

$$|C_1|^2 = \frac{96}{\pi^4}$$

$p=0$

d) $$\langle H \rangle = \sum_{p=0}^{\infty} |C_{2p+1}|^2 E_{2p+1} = \frac{96}{\pi^4}\hbar\omega_1 \sum_{p=0}^{\infty} \frac{(2p+1)^2}{(2p+1)^4}$$

$$= \frac{96}{\pi^4}\hbar\omega_1 \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} \quad , \quad \hbar\omega_1 = \frac{\hbar^2\pi^2}{2ma^2}$$

$$= \frac{96}{\pi^4}\hbar\omega_1 \frac{\pi^2}{8}$$ (GR 0.234.2)