1 Hreintóna sveifill

Gerum ráð fyrir að til sé eiginástand a_-

$$a_-|\alpha\rangle = \alpha|\alpha\rangle$$

a_- er ekki hermískur virki, því getur $\alpha \in \mathbb{C}$

a) Reiknum $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$ og $\langle p^2\rangle$ fyrir $|\alpha\rangle$

Rifjum upp

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left\{\mp ip + m\omega x\right\}$$

munum líka eftir náttúrulega lengdarskalanum

$$a = \sqrt{\frac{\hbar}{m\omega}} \quad \Longrightarrow \quad \begin{aligned} x &= \frac{a}{\sqrt{2}}\left(a_+ + a_-\right) \\ p &= \frac{i\hbar}{\sqrt{2}\,a}\left(a_+ - a_-\right) \end{aligned}$$

(2)

Því fæst

aðoka virkinn

$$\langle x \rangle = \langle \alpha | x | \alpha \rangle = \frac{a}{\sqrt{2}} \langle \alpha | \left\{ a_+ + a_- \right\} | \alpha \rangle$$

$$= \frac{a}{\sqrt{2}} \left\{ \langle a_- \alpha | \alpha \rangle + \langle \alpha | a_- | \alpha \rangle \right\} = \frac{a}{\sqrt{2}} \left\{ \alpha^* + \alpha \right\}$$

raunhluti

$$= \sqrt{2}\, a \cdot \mathrm{Re}(\alpha)$$

$$\langle p \rangle = \langle \alpha | p | \alpha \rangle = \frac{i\hbar}{\sqrt{2}\, a} \langle \alpha | \left\{ a_+ - a_- \right\} | \alpha \rangle$$

$$= \frac{i\hbar}{\sqrt{2}\, a} \left\{ \alpha^* - \alpha \right\} = -\frac{i\hbar}{\sqrt{2}\, a} \left\{ \alpha - \alpha^* \right\}$$

$$= -\frac{i\hbar \cdot 2}{\sqrt{2}\, a} i\, \mathrm{Im}(\alpha) = \sqrt{2}\, \frac{\hbar}{a}\, \mathrm{Im}(\alpha)$$

Munum og væntigildi p og x í eigin-falla grunni H hverfa

$$\langle x^2 \rangle = \frac{a^2}{2} \langle \alpha | \left[a_+ + a_- \right]^2 | \alpha \rangle = \frac{a^2}{2} \langle \alpha | \{ a_+ a_+ + a_+ a_- + a_- a_+ + a_- a_- \} | \alpha \rangle$$

Hēr er best å nota $[a_-, a_+] = 1$

$$\rightarrow \quad a_- a_+ = a_+ a_- + 1$$

$$\rightarrow \quad \langle x^2 \rangle = \frac{a^2}{2} \langle \alpha | \left[a_+ a_+ + a_+ a_- + a_+ a_- + 1 + a_- a_- \right] | \alpha \rangle$$

$$= \frac{a^2}{2} \langle \alpha | \left[(\alpha^*)^2 + 2\alpha^* \alpha + 1 + \alpha^2 \right] | \alpha \rangle$$

$$= \frac{a^2}{2} \left[(\alpha^* + \alpha)^2 + 1 \right] = \frac{a^2}{2} \left[(2\,\mathrm{Re}(\alpha))^2 + 1 \right]$$

$$\langle p^2 \rangle = -\frac{\hbar^2}{2a^2}\langle\alpha|\left\{a_+ - a_-\right\}^2|\alpha\rangle$$

$$= -\frac{\hbar^2}{2a^2}\langle\alpha|\left\{a_+a_+ - a_+a_- - a_-a_+ + a_-a_-\right\}|\alpha\rangle$$

$$= -\frac{\hbar^2}{2a^2}\left\{(\alpha^*)^2 - 2\alpha^*\alpha - 1 + \alpha^2\right\}$$

$$= \frac{\hbar^2}{2a^2}\left\{1 - (\alpha - \alpha^*)^2\right\} = \frac{\hbar^2}{2a^2}\left\{1 + (2\,\mathrm{Im}(\alpha))^2\right\}$$

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b) finna σ_x og σ_p fyrir $|\alpha\rangle$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2}{2}\left\{4(\mathrm{Re}(\alpha))^2 + 1\right\} - 2a^2(\mathrm{Re}(\alpha))^2}$$

$$= \frac{a}{\sqrt{2}}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{\hbar^2}{2a^2}\left\{1 + 4\left(\mathrm{Im}(\alpha)\right)^2\right\} - 2\frac{\hbar^2}{a^2}\left(\mathrm{Im}(\alpha)\right)^2}$$

$$= \frac{\hbar}{\sqrt{2}\,a}$$

$$\rightarrow \sigma_x \cdot \sigma_p = \frac{\hbar}{2}$$ minnsta mögulega gildið

Vitum að eiginástönd H mynda fullkomin grunn

$$\rightarrow |\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

finna c_n

$$\rightarrow \langle m|\alpha\rangle = c_m$$

því $\langle m|n\rangle = \delta_{m,n}$

En við vorum búin að finna

$$|n\rangle = A_n (a_+)^n |0\rangle \quad \text{með} \quad A_n = \frac{1}{\sqrt{n!}}$$

$$\rightarrow c_m = \langle m|\alpha\rangle = \frac{1}{\sqrt{m!}} \langle (a_+)^m 0|\alpha\rangle$$

$$= \frac{1}{\sqrt{m!}} \langle 0|(a_-)^m|\alpha\rangle = \frac{1}{\sqrt{m!}} \alpha^m \langle 0|\alpha\rangle$$

$$\rightarrow \boxed{c_m = \frac{1}{\sqrt{m!}} \alpha^m \cdot c_0}$$

d) Finnum c_0

Við vitum að $\sum_{n=0}^{\infty} |c_n|^2 = 1$

$$\rightarrow \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} |C_0|^2 = |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}$$

$$= |C_0|^2 \cdot e^{|\alpha|^2} = 1$$

$$\rightarrow |C_0|^2 = e^{-|\alpha|^2}$$

og því gæti verið

$$C_0 = e^{-\frac{|\alpha|^2}{2}}$$

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e) Hvernig er $|\alpha\rangle$ háð tíma?

$$|n(t)\rangle = |n\rangle e^{-i\omega_n t} \quad \text{með} \quad \omega_n = \frac{E_n}{\hbar} = \omega(n + 1/2)$$

því fæst

$$|\alpha(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-i\omega(n+\frac{1}{2})t} |n\rangle$$

$$= e^{-\frac{i\omega t}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} e^{-i\omega n t} |n\rangle$$

$$= e^{-\frac{i\omega t}{2}} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} |n\rangle$$

því $a|\alpha(t)\rangle$

$$= e^{-\frac{i\omega t}{2}} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} \sqrt{n}\,|n-1\rangle$$

$$= \alpha e^{-i\omega t} e^{-\frac{i\omega t}{2}} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^{n-1}}{\sqrt{(n-1)!}} e^{-\frac{|\alpha|^2}{2}} |n-1\rangle$$

$$= \alpha e^{-i\omega t} |\alpha(t)\rangle$$

þegar summuhlaupabreytunni er breytt

$(n-1) \longrightarrow m$

Ef við látum a verka á $|\alpha(t)\rangle$ fæst eigingildið

$$e^{-i\omega t}\alpha = \alpha(t)$$

$|\alpha\rangle$ og $|\alpha(t)\rangle$ eru því sama ástandið með tímaháða eigin gildið $\alpha(t) = \alpha e^{-i\omega t}$

f) $|0\rangle$ er líka svona ástand því

$a_-|0\rangle = 0$ með eigingildi 0

Rifjum aðeins upp: $\langle x\rangle = \sqrt{2}\,a\,\mathrm{Re}(\alpha)$

$$\langle x(t)\rangle = \sqrt{2}\,a\cdot\mathrm{Re}\left(\alpha e^{-i\omega t}\right)$$

Veljum α þ.a. upphafið er með α sem rauntölu

$$\rightarrow \quad \langle x(t)\rangle = \sqrt{2}\,a\,\alpha\,\cos(\omega t)$$

ástandið sveiflast fram og aftur án þess að dreifast

$$H = E\{ |1\rangle\langle 1| + 2|2\rangle\langle 2| + i|1\rangle\langle 2| - i|2\rangle\langle 1| + 3|3\rangle\langle 3| + |3\rangle\langle 1| + |1\rangle\langle 3| \}$$

Þrístiga kerfi $\{|i\rangle\}$ mynda ~~sta~~ staðlaðan grunn

Útsetning H í honum getur

$$H = E\begin{pmatrix} 1 & i & 1 \\ -i & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \langle 1|H|1\rangle & \langle 1|H|2\rangle & \langle 1|H|3\rangle \\ \langle 2|H|1\rangle & \langle 2|H|2\rangle & \langle 2|H|3\rangle \\ \langle 3|H|1\rangle & \langle 3|H|2\rangle & \langle 3|H|3\rangle \end{pmatrix}$$

Nú má finna eigingildin ~~og~~ vigrana nákvæmlega, en ég leyfi mér tölulega að ~~fara~~

þá fæst) með vigra)

$$E_1 \simeq 0.12061 \cdot E$$
$$E_2 \simeq 2.3473 \cdot E$$
$$E_3 \simeq 3.5321 \cdot E$$

$$|1) = \begin{pmatrix} a \\ ib \\ -c \end{pmatrix} \qquad |2) \simeq \begin{pmatrix} c \\ -ia \\ -b \end{pmatrix}$$

$$|3) = \begin{pmatrix} b \\ ic \\ a \end{pmatrix} \qquad \begin{aligned} a &\simeq 0.84403 \\ b &\simeq 0,44910 \\ c &\simeq 0.29313 \end{aligned}$$

Hvernig lítur H út í nýja grunninum, finna $(i|H|j)$

$|i)$ fyrir $i = 1,2$ og 3 eru eigingildi H

$\Rightarrow$ í nýja grunninum er fylki $\breve{H}$ $\begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}$

$\left\{ \text{því } (i|H|j) = (i|E_j|j) = (i|j)E_j = \delta_{ij} E_j \right\}$

Einnig er gaman að raða eiginvigrunum saman í fylki

$$V = \begin{pmatrix} a & c & b \\ ib & -ia & ic \\ -c & -b & a \end{pmatrix}$$

þá sést að $\quad V^{+}V = \mathbb{1}$

og $\quad V^{+}HV = \breve{H}$ í grunninum $\{|i)\}$

í grunninum $\{|i\rangle\}$

því er V einóta ummyndun milli grunnanna

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finnum vantigildi H í $\{|i\rangle\}$

$$\langle 1|H|1\rangle = 1E, \quad \langle 2|H|2\rangle = 2E, \quad \langle 3|H|3\rangle = 3E$$