2.51 skoðum mættið

$$V(x) = -\frac{\hbar^2 a^2}{m} \operatorname{sech}^2(ax) \quad , \quad [a] = L^{-1}$$

$[ax]$

Vídda a

$$= -2E_0 \operatorname{sech}^2(ax)$$

þar sem E_0 er einhver náttúrulegur orkustuðull mættisins

$$E_0 = \frac{\hbar^2 a^2}{2m}$$

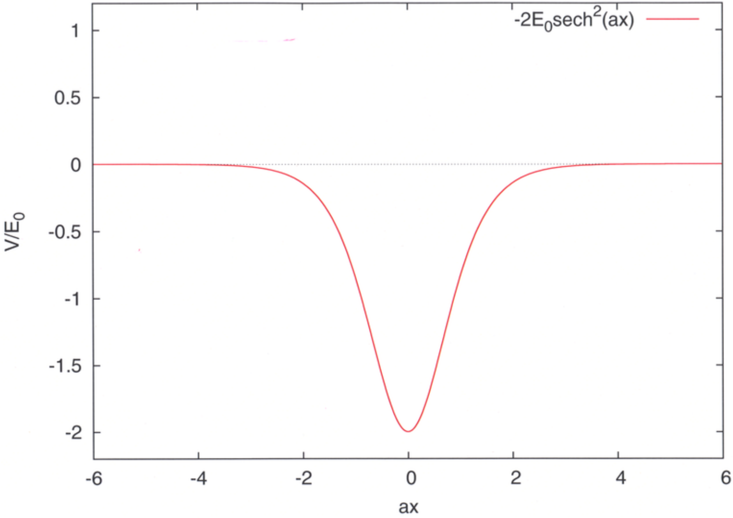
Því er eðlilegt að teikna upp V/E_0 v.s. ax

a) sjá næstu síðu

b) Sýna að $\psi_0(x) = A\operatorname{sech}(ax)$

sé bylgjufall grunnástandsins, finna A og orku grunnástandsins

2



Ég nota wxMaxima til þess að finna

$$d_x \Phi_0(x) = -Aa\,\mathrm{sech}(ax)\tanh(ax)$$
$$d_x^2 \Phi_0(x) = Aa^2\left\{\mathrm{sech}(ax)\tanh^2(ax) - \mathrm{sech}^3(ax)\right\}$$

Schrödingerjafnan fyrir grunnástandið er

$$\left\{-\frac{\hbar^2}{2m}d_x^2 - \frac{\hbar^2 a^2}{m}\mathrm{sech}^2(ax)\right\}\Phi_0(x) = \mathcal{E}_0\Phi_0(x)$$

Reynum

$$\left\{-\frac{\hbar^2}{2m}d_x^2 - \frac{\hbar^2 a^2}{m}\mathrm{sech}^2(ax)\right\}\Phi_0(x) = +\frac{\hbar^2 a^2}{m}A\left[\frac{\mathrm{sech}^3(ax) - \mathrm{sech}(ax)\tanh^2(ax)}{2} - \mathrm{sech}^3(ax)\right\}$$

④

$$= -\frac{\hbar^2 a^2}{2m}\left\{ \operatorname{Sech}^3(ax) + \operatorname{Sech}(ax)\tanh^2(ax) \right\} A$$

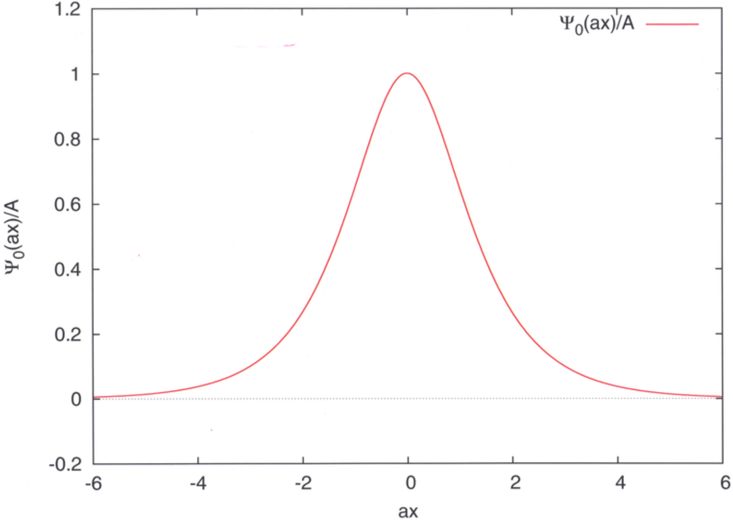
$$= -\frac{\hbar^2 a^2}{2m}\left\{ \frac{1}{\operatorname{Cosh}^3(ax)} + \frac{\operatorname{Sinh}^2(ax)}{\operatorname{Cosh}^3(ax)} \right\} A = -\frac{\hbar^2 a^2}{2m}\left\{ \frac{1+\operatorname{Sinh}^2(ax)}{\operatorname{Cosh}^3(ax)} \right\} A$$

$$= -\frac{\hbar^2 a^2}{2m} \frac{A}{\operatorname{Cosh}(ax)} = -\frac{\hbar^2 a^2}{2m} A \operatorname{sech}(ax) = -\frac{\hbar^2 a^2}{2m} \Phi_0(x)$$

$$\rightarrow \Sigma_0 = -\frac{\hbar^2 a^2}{2m} = -E_0$$

Eins og sést á næstu mynd má ~~búast við að~~ $\Phi_0(x)$ sé bylgjufall grunnástandsins, engin núllstöð, samkvæmt.

5



Stoðla $\psi_0 = A\,\mathrm{sech}(ax) = A\frac{1}{\cosh(ax)}$

$$1 = |A|^2 \int_{-\infty}^{\infty} dx\, \mathrm{sech}^2(ax) = \frac{a}{2}$$

samkvæmt wXmaxima

$\rightarrow A = \sqrt{\frac{a}{2}}$ er mögulegt val fyrir A.

c) Sýna að

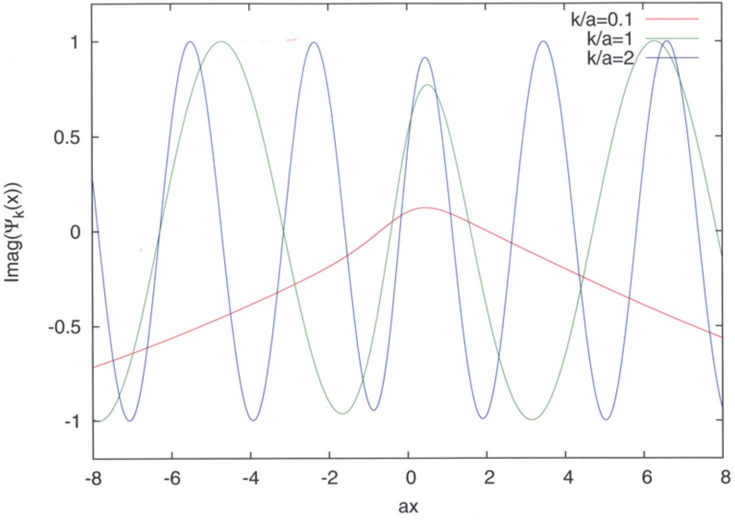
$$\psi_k(x) = A\left\{\frac{ik - a\tanh(ax)}{ik + a}\right\}e^{ikx}$$

með $k = \frac{\sqrt{2mE}}{\hbar}$ sé lausn á jöfnu Schrödingers

fyrir hvaða E sem er.

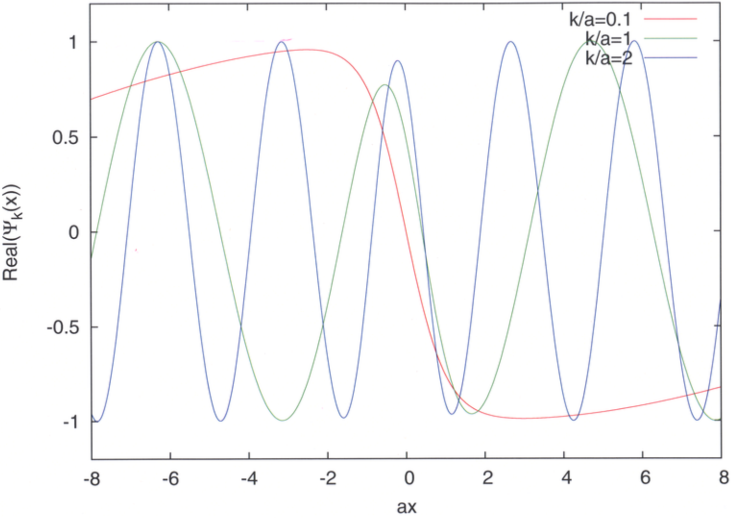
$$\psi_k(x) = A\left\{\frac{i\left(\frac{k}{a}\right) - \tanh(ax)}{i\left(\frac{k}{a}\right) + 1}\right\}e^{i\left(\frac{k}{a}\right)(ax)}$$

Skoðum graf fyrst

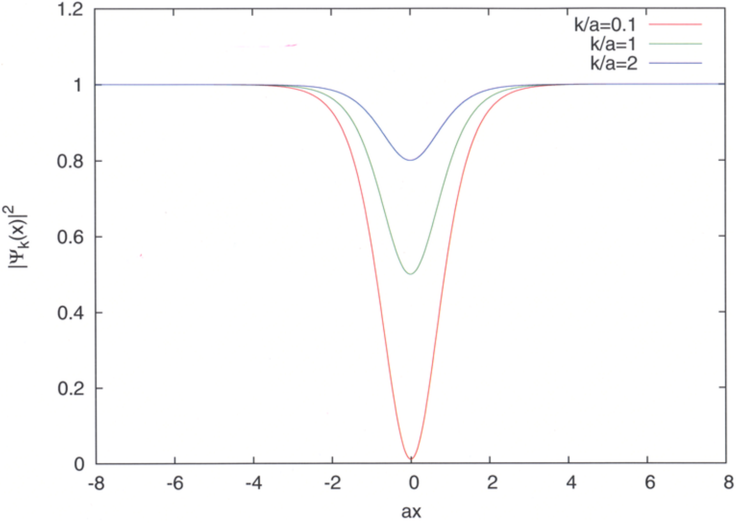


k/a=0.1
k/a=1
k/a=2
1
0.5
0
-0.5
-1
Imag(Ψ_k(x))
-8
-6
-4
-2
0
2
4
6
8
ax

⑨



②



Reynum

$$H\psi_k(x) = A\left\{-\frac{\hbar^2}{2m}d_x^2 - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\right\}\left\{\frac{ik - a\tanh(ax)}{ik+a}\right\}e^{ikx}$$

$$d_x^2\psi_k(x) = A\left[-\frac{k^2 e^{ikx}(ik - a\tanh(ax))}{ik+a} + \frac{2a^3 e^{ikx}\operatorname{sech}^2(ax)\tanh(ax)}{ik+a} - \frac{2ia^2 k e^{ikx}\operatorname{sech}^2(ax)}{ik+a}\right]$$

$$H\psi_k(x) = -\frac{A\hbar^2}{2m}\left[k^2 e^{ikx}\left\{\frac{ik - a\tanh(ax)}{ik+a}\right\} + \frac{2a^3 e^{ikx}\operatorname{sech}^2\cancel{(ax)}\tanh(ax)}{\cancel{ik}+a} - \frac{2ia^2 k e^{ikx}\operatorname{s}\cancel{ech}^2(ax)}{ik+a} + 2a^2\cancel{\operatorname{sech}}^2(ax)\left\{\frac{ik - a\cancel{\tanh}(ax)}{\cancel{ik}+a}\right\}e^{ikx}\right]$$

$$H\psi_k(x) = -\frac{\hbar^2 k^2}{2m} A e^{ikx} \left\{ \frac{ik - a\tanh(ax)}{ik+a} \right\} = \underbrace{-\frac{\hbar^2 k^2}{2m}}_{\text{orka frjálsrar eindar}} \psi_k(x)$$

$$\left.\begin{aligned} \lim_{x\to\infty} \tanh(x) &= +1 \\ \lim_{x\to-\infty} \tanh(x) &= -1 \end{aligned}\right\}$$

Aðfellu form innbylgju er

$$\psi_k(x) \xrightarrow[x\to-\infty]{} A e^{ikx}$$

Aðfellu form út bylgju er

$$\psi_k(x) \xrightarrow[x\to\infty]{} A\left\{\frac{ik-a}{ik+a}\right\} e^{ikx} \to \underbrace{|\psi_k(x)|^2 \xrightarrow[x\to\pm\infty]{} A}_{\text{engin endurkast bylgja!}}$$

Nú er hægt að spyrja eru til önnur svona mætti?
Einfaldari spurning er: Er til mætti $V(x,\alpha)$ með stika α þ.a. eigingildi jöfnu Schrödingers breytist ekki þegar α er hnikað til. Hægt er að sýna að mætti sem uppfylla ólínulegu jöfnuna {Korteweg-deVries}

$$\partial_\alpha V + V[\partial_x V] + \partial_x^3 V = 0$$

stika þannig mættum. → Lausnir á öfugadreifivandamáli fyrir línulegu jöfnu Schrödinger leiða til lausna á ólínulegum jöfnum .... einfara lausnir ....

2.45

Engar margfaldar lausnir í einvídd fyrir bundin ástönd

Gefum okkur að til séu tvær lausnir ψ_1 og ψ_2 með sömu orku E

$$\left\{-\frac{\hbar^2}{2m}d_x^2 + V(x)\right\}\psi_1 = E\psi_1 \qquad \text{margfalda með } \psi_2$$

$$\left\{-\frac{\hbar^2}{2m}d_x^2 + V(x)\right\}\psi_2 = E\psi_2 \qquad -||- \quad \psi_1$$

draga frá

$$\rightarrow \quad -\psi_2 \frac{\hbar^2}{2m}\psi_1'' + \psi_1 \frac{\hbar^2}{2m}\psi_2'' = E\left\{\psi_2\psi_1 - \psi_1\psi_2\right\} = 0$$

$$\rightarrow \quad \psi_1\psi_2'' - \psi_2\psi_1'' = 0$$

Athugum

$$d_x\left\{\psi_1 d_x\psi_2 - \psi_2 d_x\psi_1\right\} = \left\{\psi_1 d_x^2\psi_2 - \psi_2 d_x^2\psi_1\right\} + \left\{\psi_1'\psi_2' - \psi_2'\psi_1'\right\}$$

$$= \left\{\psi_1\psi_2'' - \psi_2\psi_1''\right\} = 0 \longleftarrow \text{eins og sást áður}$$

$$\rightarrow \left\{\psi_1\psi_2' - \psi_2\psi_1'\right\} = C \longleftarrow \text{fasti}$$

Bundin ástönd $\rightarrow$ $\psi_1(x)$, $\psi_2(x)$ $\xrightarrow[x\to\pm\infty]{} 0$ $\rightarrow$ $C = 0$

$$\rightarrow \left\{ \psi_1 \psi_2' - \psi_2 \psi_1' \right\} = 0$$ fyrir öll x

$$\rightarrow \psi_1 \psi_2' = \psi_2 \psi_1' \rightarrow \frac{\psi_2'}{\psi_2} = \frac{\psi_1'}{\psi_1}$$

$$\rightarrow \ln \psi_1 = \ln \psi_2 + C_1$$

$$\rightarrow \boxed{\psi_1 = C_1 \psi_2}$$

sama bylgjufallið