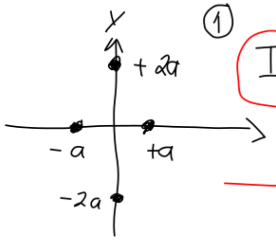
Dæmi 1

Massi m í
(+a,0,0) og (-a,0,0),
(0,+2a,0) og (0,-2a,0)



$$\boxed{I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha i} x_{\alpha j}\right]}$$

$$\boxed{= ma^2 \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{pmatrix}}$$

② Festum

$$\bar{\omega} = \frac{\omega}{\sqrt{3}}(1,1,1)$$

$$\longrightarrow \boxed{L = I \cdot \bar{\omega}} = ma^2 \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{\omega}{\sqrt{3}}$$

$$\boxed{= \frac{ma^2\omega}{\sqrt{3}}(8,2,10)}$$

② Setjum $\hat{n} = \frac{1}{\sqrt{3}}(1,1,1)$

$$L \cdot \hat{n} = ma^2 \frac{\omega}{3} 20 = |L| \cos\gamma$$

$$|L| = \omega m a^2 \frac{1}{\sqrt{3}} \sqrt{64 + 4 + 100} \simeq ma^2\omega \, 7{,}4833$$

$$\longrightarrow \cos\gamma = \frac{L \cdot \hat{n}}{|L|} \simeq 0{,}89087 \longrightarrow \boxed{\gamma = 0{,}4715 \text{ rad} \simeq 27^\circ}$$

þar sem γ er hornið milli vigranna. Mikilvægt er að muna að hér er allt reiknað í hnitakerfi hlutar. Sú aðferðafræði einfaldar alla reikningana. Það eru ein mikilvægustu skilboðin úr köflunum um snúningshreyfingu og hnitakerfi sem ekki eru tregðukerfi.

Dæmi 2

Athugum samhverfan snúð án ytra kraftvægis eins og fjallað er um í undirkafla 13.20.1 í bók DC. Umfjöllunin endar með jöfnu (13.139) fyrir tímaafleiðu Eulerhornsins φ, en bein lausn sem fall af tímanum t er sjaldan sett fram fyrir hornin þrjú. Gerum ráð fyrir að snúðurinn sé á hreyfingu með jafnri veltu sem lýst er með jöfnu (13.119). Finnið tímaþróun allra hornanna.

Snúður án ytra vægis $\longrightarrow$

$$\begin{cases} (I_1 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0, & I_1 = I_2 \\ (I_3 - I_1)\omega_3\omega_1 - I_1\dot{\omega}_2 = 0 \end{cases}$$

$$I_3\dot{\omega}_3 = 0 \longrightarrow \underline{\omega_3 = \text{fasti}}$$

$$\begin{matrix} \dot{\omega}_1 + \Omega\omega_2 = 0 \\ \dot{\omega}_2 - \Omega\omega_1 = 0 \end{matrix} \quad (*)$$

$$\underset{\uparrow \text{fasti}}{\Omega} = (-\dot{\psi}) = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3$$

Höfum (13.86-88)

$$\begin{cases} \omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \end{cases} \qquad (z)$$

notum með (*) $\longrightarrow$

til að uppfylla upphafsgildi

$$\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi = A\cos(\Omega t + \gamma) \qquad (a)$$

$$\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = A\sin(\Omega t + \gamma) \qquad (b)$$

$$\dot{\phi}\cos\theta + \dot{\psi} = \text{fasti} \equiv B$$

(5)

Notum (a)·cos ψ og (b)·sin ψ →

$$\dot\phi \sin\theta \sin\psi\cos\psi + \dot\theta \cos^2\psi = A\cos(\Omega t+\gamma)\cos\psi \qquad (1)$$

$$\dot\phi \sin\theta \cos\psi\sin\psi - \dot\theta \sin^2\psi = A\sin(\Omega t+\gamma)\sin\psi \qquad (2)$$

(1) − (2):

$$\dot\theta = A\cos(\Omega t+\gamma)\cos\psi - A\sin(\Omega t+\gamma)\sin\psi$$

$$= A\cos(\Omega t+\gamma+\psi)$$

jöfn velta → $\dot\theta = 0$ → $A\cos(\Omega t+\gamma+\psi) = 0$

$$\to \Omega t + \gamma + \psi = \left(\frac{2n+1}{2}\right)\pi , \quad n = 0,1,2,3,\ldots$$

(6)

$$\to \boxed{\psi = \psi_0 - \Omega t} , \quad \psi_0 = -\gamma + \frac{(2n+1)\pi}{2}$$ upphafsgildi

(a)·sin ψ + (b) cos ψ →

$$\dot\phi \sin\theta \sin\psi \sin\psi = A\cos(\Omega t+\gamma)\sin\psi$$

$$+$$

$$\dot\phi \sin\theta \cos\psi\cos\psi = A\sin(\Omega t+\gamma)\cos\psi$$

$$\to \quad \dot\phi \sin\theta = A\sin(\Omega t+\gamma+\psi)$$

$$\to \quad \dot\phi = \frac{A}{\sin\theta}\sin(\Omega t+\gamma+\psi) \qquad (***)$$

(7)

notum í (2) og $\psi = \psi_0 - \Omega t$ → $\dot\psi = -\Omega$

$$\to \omega_3 = A\frac{\cos\theta}{\sin\theta}\sin(\Omega t+\gamma+\psi) - \Omega$$

$$= A\frac{\cos\theta}{\sin\theta}\sin(\Omega t+\gamma+\psi_0-\Omega t) - \Omega$$

$$\to \omega_3 + \Omega = \frac{A}{\tan\theta}\sin(\gamma+\psi_0) = \frac{A}{\tan\theta}\sin\left(\frac{(2n+1)\pi}{2}\right)$$

$$= \frac{A}{\tan\theta}$$

$$\to \boxed{A = (\omega_3+\Omega)\tan\theta}$$

(8)

$$A = \left[\omega_3 + \left(\frac{I_3 - I_1}{I_1}\right)\omega_3\right]\tan\theta = \frac{I_3}{I_1}\tan\theta$$

$$(***) \to \quad \dot\phi = \frac{I_3}{I_1}\frac{\tan\theta}{\sin\theta}\sin(\Omega t+\gamma+\psi)$$

$$= \frac{I_3}{I_1}\frac{1}{\cos\theta}\sin(\underbrace{\gamma+\psi_0}_{\frac{2n+1}{2}\pi})$$

$$= \frac{I_3}{I_1}\frac{1}{\cos\theta}$$

$$\to \boxed{\phi = \phi_0 + \frac{I_3 t}{I_1\cos\theta}}$$

virðist löng leið fyrir svör með einfalt útlit

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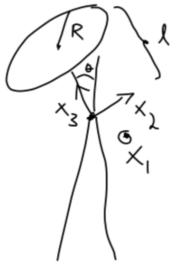
Dæmi 3

veltuhraði $\bar{\omega} = \Omega \hat{z}$

Steiner

$$I_3 = \frac{MR^2}{2}, \quad I_1 = I_2 = I = \frac{MR^2}{4} + Ml^2$$



$$\underbrace{\omega_3 = \Omega\cos\theta \qquad \omega_2 = \Omega\sin\theta}$$

$$L = \mathbb{I}\cdot\bar{\omega}$$

$$L = I_3\Omega\cos\theta\,\hat{x}_3 + I\Omega\sin\theta\,\hat{x}_2$$

(13.110) með

$I_1 = I_2$, $\dot{\omega}_3 = 0$

$\omega_1 = 0$, $\dot{\omega}_2 = 0$

$$\underbrace{I\dot{\omega}_1}_{=0} - (I - I_3)\omega_2\omega_3 = N_1$$

$$\rightarrow -(I - I_3)\omega_2\omega_3 = N_1$$

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$$-(I - I_3)\Omega^2\cos\theta\sin\theta = N_1 = Mgl\sin\theta$$

$$\rightarrow -\left(\frac{MR^2}{4} + Ml^2 - \frac{MR^2}{2}\right)\Omega^2\cos\theta = Mgl$$

$$\rightarrow \left(\frac{MR^2}{4} - Ml^2\right)\Omega^2\cos\theta = Mgl$$

$$\rightarrow \Omega^2 = \frac{Mgl}{\cos\theta}\,\frac{1}{\left(\frac{MR^2}{4} - Ml^2\right)} = \frac{4gl}{\cos\theta\,(R^2 - 4l^2)}$$

$$\rightarrow \boxed{\Omega = \sqrt{\frac{4gl}{\cos\theta\,(R^2 - 4l^2)}}} \quad \text{ef} \quad \boxed{R > 2l}$$

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Dæmi 4

Lítum á kartískt hnitakerfi. Notið horn Eulers til að finna eitt mögulegt val fyrir snúning frá (1,0,0) yfir í einingavigurinn n sem liggur mitt á milli jákvæðu hluta x-, y- og z-ásanna. Finnið snúningsfylkin og sannreynið niðurstöðuna. Hvert er hornið frá n niður á hvern ásanna x, y, eða z?

Frá DC jöfnu (13.83) fáum við   því $n = (1,1,1)/\sqrt{3}$

$$\cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi = \frac{1}{\sqrt{3}}$$

$$-\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi = \frac{1}{\sqrt{3}}$$

$$\sin\phi\sin\theta = \frac{1}{\sqrt{3}}$$

Við höfum nokkuð frelsi, svo ég vel að reyna $\phi = -\frac{\pi}{4}$

$$\rightarrow \begin{cases} \cos\psi + \cos\theta\sin\psi = \sqrt{\frac{2}{3}} \\ -\sin\psi + \cos\theta\cos\psi = \sqrt{\frac{2}{3}} \end{cases} \quad \text{og} \quad \sin\theta = -\sqrt{\frac{2}{3}}$$

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$$\rightarrow \cos\theta = \sqrt{1 - \frac{2}{3}} = \sqrt{\frac{1}{3}}$$

og

$$\left.\begin{array}{l} \cos\psi + \frac{1}{\sqrt{3}}\sin\psi = \sqrt{\frac{2}{3}} \\ -\sin\psi + \frac{1}{\sqrt{3}}\cos\psi = \sqrt{\frac{2}{3}} \end{array}\right\} \rightarrow \begin{array}{l} \cos\psi = \frac{1+\sqrt{3}}{2} \\ \sin\psi = \frac{1-\sqrt{3}}{2} \end{array}$$

$$\lambda_\phi = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \lambda_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}$$

$$\lambda_\psi = \begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{2}} & 0 \\ -\frac{1-\sqrt{3}}{2\sqrt{2}} & \frac{1+\sqrt{3}}{2\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{\lambda_\psi\lambda_\theta\lambda_\phi\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$\cos\gamma = \frac{1}{\sqrt{3}}$$

hornið niður á ás, $\gamma \sim 54{,}7^\circ$