(1)

Dæmi 1

M í a(1,0,1)
M í a(0,2,0)
2M í a(0,-2,0)
M í a(1,0,0)

$$I_{ij} = \sum_{\alpha} m_\alpha \left[\delta_{ij} \sum_{k} x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right]$$

$$\rightarrow \boxed{I = Ma^2 \begin{pmatrix} 13 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 14 \end{pmatrix}}$$

→ eigingildin eru, og vigrarnir →

$$Ma^2\left(\frac{27-\sqrt{5}}{2},\ 3,\ \frac{27+\sqrt{5}}{2}\right)$$

$$\left.\begin{matrix} \frac{\sqrt{2}a}{\sqrt{5-\sqrt{5}}}\left(1,0,\frac{\sqrt{5}-1}{2}\right) \\ a(0,1,0) \\ \frac{\sqrt{2}a}{\sqrt{5+\sqrt{5}}}\left(1,0,-\frac{\sqrt{5}+1}{2}\right) \end{matrix}\right\} \text{höfuðásar}$$

(2)

Útbúum S með því að setja eiginvigrana sem dálka

$$S \doteq \begin{pmatrix} 0{,}85065 & 0 & 0{,}52573 \\ 0 & 1 & 0 \\ 0{,}52573 & 0 & -0{,}85065 \end{pmatrix}$$

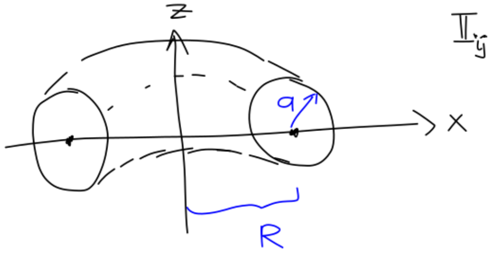
Hornalínuhamur

$$\rightarrow S \cdot I \cdot S^t \sim \begin{pmatrix} 12{,}382 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14{,}618 \end{pmatrix} \sim I_{diag.}$$

í wxmaxima: "S . I . transpose(S);", þar sem bilin um punktana tákna fylkjamarföldun

(3)

Dæmi 2 Kleinuhringur



$$I_{ij} = \int_V dV \rho(\vec{r}) \left\{\delta_{ij} \sum_k x_k^2 - x_i x_j\right\}$$

Ég vel sívalningshnit, en það þarf að passa sig

$$\left\{\begin{aligned} \sum_k x_k^2 &= x^2+y^2+z^2 \quad \text{kartísk} \\ &= r^2+z^2 \quad \text{sívaln.} \end{aligned}\right.$$

$$I_{33} = \rho \int_V dV (r^2+z^2-z^2) = \rho \int_V dV\, r^2$$

$$= 2\rho \int_0^{2\pi} d\theta \int_{R-a}^{R+a} r\,dr \int_0^{\sqrt{a^2-(r-R)^2}} dz$$

Ég tek eftir að heildunin í z er samhverf um láréttusléttuna og nota jöfnu hrings miðjaðs í R sem efri mörk. Þá er r-ið takmarkað frá R-a upp í R+a

(4)

$$\rightarrow I_{33} = 2\pi 2\rho \int_{R-a}^{R+a} r^3 dr \sqrt{a^2-(r-R)^2} = 4\pi\rho\left[\frac{3\pi R a^4 + 4\pi R^3 a^2}{8}\right]$$

hér var mjög þægilegt að nota wxmaxima og setja inn mörkin í heildið

$$= \frac{\pi^2 \rho R a^2}{2}\left\{3a^2 + 4R^2\right\}$$

$$V = 2\pi R \cdot \pi a^2 = 2\pi^2 R a^2$$

$$\rightarrow \rho = \frac{M}{2\pi^2 R a^2}$$

$$\rightarrow \boxed{I_{33} = \frac{\pi^2 M R a^2}{2 \cdot 2\pi^2 R a^2}\left\{3a^2+4R^2\right\} = M\left[\frac{3a^2}{4} + R^2\right]}$$

$$I_{11} = \rho \int_V dv \left\{x^2+y^2+z^2-x^2\right\} = \rho \int_V dv \left\{y^2+z^2\right\}$$

$$= 2\rho \int_0^{2\pi} d\theta \int_{R-a}^{R+a} r\,dr \int_0^{\sqrt{a^2-(r-R)^2}} dz\left(r^2 \sin^2\theta + z^2\right)$$

$$= 2\rho \int_0^{2\pi} d\theta \int_{R-a}^{R+a} r\,dr \left\{ z r^2 \sin^2\theta + \frac{z^3}{3}\right\}\Bigg|_0^{\sqrt{a^2-(r-R)^2}}$$

$$I_{11} = 2\rho \int_0^{2\pi} d\theta \int_{R-a}^{R+a} r\,dr \left\{ \sqrt{a^2-(r-R)^2}\, r^2 \sin^2\theta + \frac{(a^2-(r-R)^2)^{3/2}}{3}\right\}$$

$$= 2\rho \int_0^{2\pi} d\theta \left[\sin^2\theta\, \frac{3\pi R a^4 + 4\pi R^3 a^2}{8} + \frac{\pi R a^4}{8}\right]$$

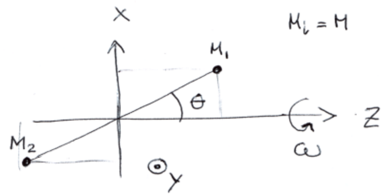
$$= 2\rho \left\{ \frac{3\pi^2 R a^4 + 4\pi^2 R^3 a^2}{8} + \frac{2\pi^2 R a^4}{8}\right\}$$

$$= 2\rho\left[\frac{5\pi^2 R a^4}{8} + \frac{4\pi^2 R^3 a^2}{8}\right] = \rho\, 2\pi^2 R a^2 \left(\frac{5a^2}{8} + \frac{4R^2}{8}\right)$$

$$\to \boxed{I_{11} = M\left[\frac{5a^2}{8} + \frac{R^2}{2}\right]}$$

Sama niðurstaða fæst fyrir I_{22} og einfalt er að sannreyna að allir aðrir þættir þinsins hverfa vegna samhverfu, enda er víst að þessir samhverfuásar séu höfuðásar kleinuhringsins.

Dæmi 3



Þyrill í eigin hnitakerfi

$M_1 = M$ í $A(\sin\theta, 0, \cos\theta)$

$M_2 = M$ í $-A(\sin\theta, 0, \cos\theta)$

$\bar{\omega} = \omega\, \hat{e}_z$

Snúningur um CM (0-punkt eigin hnitakerfis)

$$I = MA^2 \begin{pmatrix} 2\cos^2\theta & 0 & -2\sin\theta\cos\theta \\ 0 & 2 & 0 \\ -2\sin\theta\cos\theta & 0 & 2\sin^2\theta \end{pmatrix}$$

$$\bar{L} = I \cdot \bar{\omega}$$

$$= 2MA^2 \begin{pmatrix} \cos^2\theta & 0 & -\sin\theta\cos\theta \\ 0 & 1 & 0 \\ -2\sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

$$= 2MA^2\omega \begin{pmatrix} -\sin\theta\cos\theta \\ 0 \\ \sin^2\theta \end{pmatrix} = 2MA^2\omega\sin\theta \begin{pmatrix} -\cos\theta \\ 0 \\ \sin\theta \end{pmatrix}$$

berum saman við

$$\bar{L} = \sum_\alpha m_\alpha (\bar{r}_\alpha \times \bar{v}_\alpha)$$

$$\bar{r}_1 = A(\sin\theta, 0, \cos\theta)$$

$$\bar{r}_2 = -\bar{r}_1$$

$$\bar{v}_1 = \bar{\omega} \times \bar{r}_1 = A\sin\theta\cdot\omega\,\hat{e}_y$$

$$\bar{v}_2 = \bar{\omega} \times \bar{r}_2 = -A\sin\theta\cdot\omega\,\hat{e}_y$$

$$\bar{L} = 2MA^2\omega\sin\theta \begin{pmatrix} -\cos\theta \\ 0 \\ \sin\theta \end{pmatrix}$$

$$T_{rot} = \frac{1}{2}\bar{\omega}\cdot\mathbb{I}\cdot\bar{\omega}$$

$$= (2MA^2)\frac{1}{2}(0,0,\omega) \begin{pmatrix} \cos^2\theta & 0 & -\sin\theta\cos\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

$$= \frac{1}{2} 2MA^2\omega\sin\theta\,(0,0,\omega) \begin{pmatrix} -\cos\theta \\ 0 \\ \sin\theta \end{pmatrix}$$

$$= \frac{1}{2} 2MA^2\omega^2\sin^2\theta = M(A\omega\sin\theta)^2$$

$$\bar{N} = \bar{\omega} \times \bar{L} = (0, -L_x\omega, 0) = 2MA^2\omega^2\sin\theta\cos\theta\,\hat{e}_y$$

Dæmi 4

4M í a(0,0,1)
1M í a(0,1,0)
1M í a(1,0,0)

$$\rightarrow \bar{r}_{CM} = \frac{aM}{6M}\left[4(0,0,1) + (0,1,0) + (1,0,0)\right]$$

$$= a\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$$ hnit massamiðjunnar

$$\mathbb{J} = Ma^2 \begin{pmatrix} 4+1 & 0 & 0 \\ 0 & 4+1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

hverfitregðuþinurinn fyrir upphafspunkt hnitakerfisins massa-kerfisins

Fyrir snúning um massamiðjuna:

$$\mathbb{I}^{CM}_{ij} = \mathbb{J}_{ij} - 6M\left((r_{CM})^2\delta_{ij} - (\bar{r}_{CM})_i(\bar{r}_{CM})_j\right)$$

$$= \mathbb{J}_{ij} - 6Ma^2 \begin{pmatrix} \frac{1}{36}+\frac{4}{9} & -\frac{1}{36} & -\frac{2}{18} \\ -\frac{1}{36} & \frac{1}{36}+\frac{4}{9} & -\frac{2}{18} \\ -\frac{2}{18} & -\frac{2}{18} & \frac{2}{36} \end{pmatrix}$$

$$= Ma^2 \begin{pmatrix} \frac{13}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{13}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{5}{3} \end{pmatrix}$$