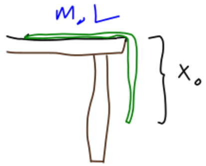
1

Dæmi 1   Lengd L, $x(0)=x_0$, $\dot{x}(0)=0$, $x_0>0$



Gamla lausnin var

$$\ddot{x} - \frac{g}{L}x = 0 \,, \quad \tau = \sqrt{\frac{L}{g}}\,\mathrm{ArCosh}\left(\frac{L}{x_0}\right)$$

τ er tíminn þar til reipið fer út af borðinu

Notum núna orkuna til að nálgast lausnina

$$T = \frac{m}{2}\dot{x}^2 \,, \quad F(x) = mx\frac{g}{L}$$

þyngdarkrafturinn á reipið þegar x hangir fram af borðinu, því hluti massans sem hangir er m(x/L)

$$-\frac{\partial U}{\partial x} = F$$

$$\hookrightarrow dU = -F\,dx$$

$$\int_{U(x_0)}^{U(x)} dU = -\frac{mg}{L}\int_{x_0}^{x} dx'\,x' \;\to\; U(x) - U(x_0) = -\frac{mg}{2L}\left(x^2 - x_0^2\right)$$

2

setjum $E_T = U(x_0) = 0$ heildarorkan

$$E_T = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + U(x) = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 - \frac{mg}{2L}\left(x^2 - x_0^2\right)$$

$$\to \left(\frac{dx}{dt}\right)^2 = \frac{g}{L}\left(x^2 - x_0^2\right) \;\to\; dt = \pm\frac{dx}{\sqrt{\frac{g}{L}}\sqrt{x^2 - x_0^2}}$$

$$\to \int_0^{\tau} dt = \pm\sqrt{\frac{L}{g}}\int_{x_0}^{L}\frac{dx}{\sqrt{x^2 - x_0^2}} = \pm\sqrt{\frac{L}{g}}\left\{\ln\left[2\sqrt{x^2 - x_0^2} + 2x\right]\Big|_{x_0}^{L}\right\}$$

$$= \pm\sqrt{\frac{L}{g}}\left\{\ln\left[2\sqrt{L^2 - x_0^2} + 2L\right] - \ln(2x_0)\right\}$$

$$= \pm\sqrt{\frac{L}{g}}\left[\ln\left[\sqrt{\left(\frac{L}{x_0}\right)^2 - 1} + \frac{L}{x_0}\right]\right]$$

3

Gradsteyn (1.622.6) eða wikipedia gefa okkur að

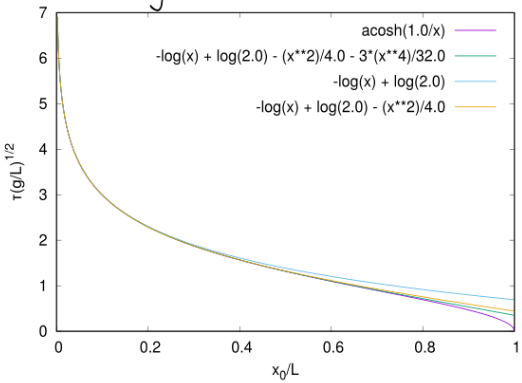
$$\mathrm{ArCosh}(z) = \ln\left[z + \sqrt{z^2 - 1}\right]$$

og því

$$\tau = \sqrt{\frac{L}{g}}\,\mathrm{ArCosh}\left(\frac{L}{x_0}\right)$$

eins og áður, og við völdum jákvæðu grein rótarinnar

skoðum mynd



Hér sést að tíminn lengist upp úr öllu valdi þegar x_0 --> 0, skoðum þetta nánar

4

$$\mathrm{ArCosh}\left(\frac{1}{z}\right) = \ln\left[\frac{1}{z} + \sqrt{\frac{1}{z^2} - 1}\right] = \ln\left[\frac{1}{z} + \frac{1}{z}\sqrt{1 - z^2}\right]$$

$$= \ln\left[1 + \sqrt{1 - z^2}\right] - \ln(z) \approx \ln\left[1 + 1 - \frac{z^2}{2} - \frac{z^4}{8}\dots\right] - \ln(z)$$

$$= \ln\left[2 - \frac{z^2}{2} - \frac{z^4}{8}\right] - \ln(z)$$

$$= \ln(2) + \ln\left[1 - \frac{z^2}{4} - \frac{z^4}{16}\right] - \ln(z)$$

$$\approx \ln(2) - \frac{z^2}{4} - \ln(z)$$ þegar z --> 0

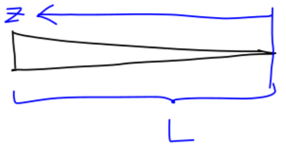
Eins gefur wxmaxima

$$\mathrm{ArCos}\left(\frac{1}{z}\right) \to -\ln(z) + \ln(2) - \frac{x^2}{4} - \frac{3x^4}{32} + \dots$$

Ég setti því nálganir fyrir tímann á grafið á síðunni á undan

5

Dæmi 2 Reipi



þéttleikinn í z $\lambda(z) = \alpha z$

Togum reipið upp á þunna endanum finnum fallið sem lýsir massanum sem kominn er af borðinu þegar hæð endans er z yfir því

$$M(z) = \int_0^z dz' \, \lambda(z') = \frac{\alpha z^2}{2}, \quad M(L) = M = \frac{\alpha L^2}{2}$$

$$\to \alpha = \frac{2M}{L^2} \to M(z) = \frac{M}{L^2} z^2$$

þyngd reipis á lofti $M(z)$

Átlag á hönd

$$F_{ump} = \frac{d}{dt}\left\{M(z) v\right\}, \quad v = at$$

$$= \frac{d}{dt}\left\{\frac{M}{L^2} z^2 v\right\} = \frac{M}{L^2} a \frac{d}{dt}\left\{z^2 t\right\}$$

6

$$\to F_{ump} = \frac{M}{L^2} a \left[2z\left(\frac{dz}{dt}\right) t + z^2\right] = \frac{M}{L^2} a \left[2z \dot{z} t + z^2\right]$$

$$= \frac{Ma}{L^2}\left[2z v t + z^2\right] = \frac{M}{L^2}\left[2z v^2 + z^2 a\right]$$

..en,

$$\ddot{z} = \frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz} = a$$

$$\to v\,dv = a\,dz \to v^2 = 2az$$

$$\to F_{ump} = \frac{M}{L^2}\left[2z\, 2az + z^2 a\right] = \frac{M}{L^2} 5 z^2 a$$

$$\to F_T = M(z) g + \frac{M}{L^2} 5 z^2 a = \frac{M}{L^2} z^2 \left[g + 5a\right]$$

7

eða, merkilegt nokk:

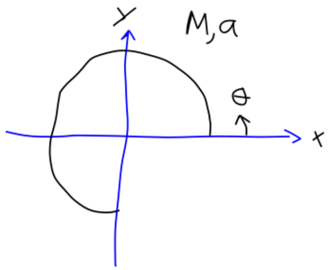
$$F_T = M(z)\left[g + 5a\right]$$

heildarkrafturinn á höndina

Nú, vaknar spurningin: Hvað með svona reipi í fyrsta dæmi? Ráðum við við að lýsa því. Svarið er já, en hreyfijafnan verður þannig að best er að leysa hana tölulega, og ef við notum aðferðina með heildarorkunni endum við með ellípsk heildi, ekki af einföldustu gerð!

8

Dæmi 3 Setjum 3/4 sneið pizzu upp í hnitakerfi



$x = r\cos\theta$

$y = r\sin\theta$

$\sigma = \frac{M}{A} \to M = \sigma A$

Massi pizzu á flöt

Flötur pizzu

$$\bar{R}_{cm} = \frac{1}{M}\int_A \bar{r}\, dM = \frac{1}{\sigma A}\int_A \bar{r}\sigma\, dA = \frac{1}{A}\int_A \bar{r}\, dA$$

$$A = (\pi a^2)\frac{3}{4}$$

Hér játast að A hefur fleiri ein eitt hlutverk

(a)

$$\boxed{\bar{R}_{CM}} = \frac{1}{A}\int_0^a r\,dr\int_0^{3\pi/2} d\theta\,(r\cos\theta, r\sin\theta)$$

$$= \frac{1}{A}\int_0^a r^2 dr\int_0^{3\pi/2} d\theta\,(\cos\theta,\sin\theta) = \frac{a^3}{3A}(-1,1)$$

$$= \frac{4a^3}{3\pi a^2\, 3}(-1,1) = \boxed{\frac{4a}{9\pi}(-1,1)}$$

$$\simeq 0{,}1415\, a\,(-1,1)$$

Kemur þetta á óvart, eða er í samræmi við tilfinningu ykkar?

(b)

Dæmi 4

Setjum 270° bauginn með massa M og geisla a eins upp í pólhnitum

$$\lambda = \frac{M}{L}, \quad L = 2\pi a\cdot\frac{3}{4}$$

$$\boxed{\bar{R}_{CM}} = \frac{1}{M}\int \bar{r}\,dM = \frac{1}{\lambda L}\int \bar{r}\,\lambda\,dL$$

$$= \frac{1}{L}\int \bar{r}\,a\,d\theta = \frac{a\cdot a}{L}\int_0^{3\pi/2} d\theta\,(\cos\theta,\sin\theta)$$

$$= \frac{a^2}{L}(-1,1) = \frac{4a^2}{6\pi a}(-1,1) = \boxed{\frac{2a}{3\pi}(-1,1)}$$

$$\simeq 0{,}2122\, a\,(-1,1)$$