(1)

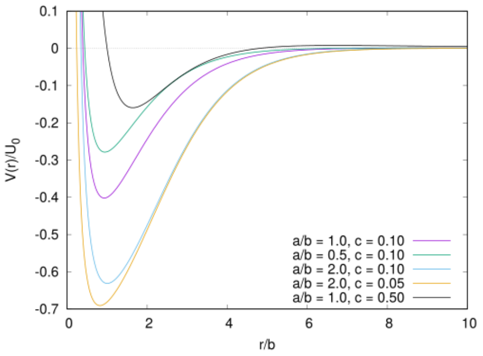
Dæmi 1 Wood-Saxon

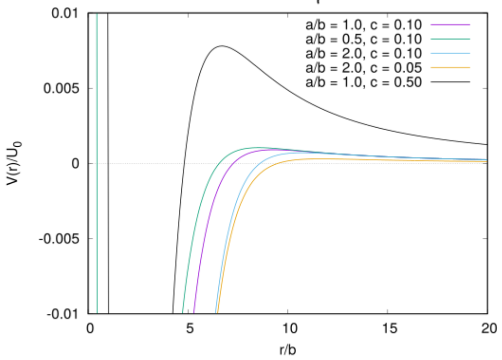
$$U(r) = -\frac{U_0}{1+\exp\left[\frac{r-a}{b}\right]}, \quad a,b>0$$

(1) Virka mættið

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2}, \quad \ell = \mu r^2\dot{\theta} \ \text{fasti}$$

(2) Geisli hringbrautar, teiknum fyrst virka mættið $c = \frac{\ell^2}{2\mu b^2}$




(2)

Búumst við að ögnin geti verið á hringbraut í lágmarki fyrir lágan hverfiþunga skoðum

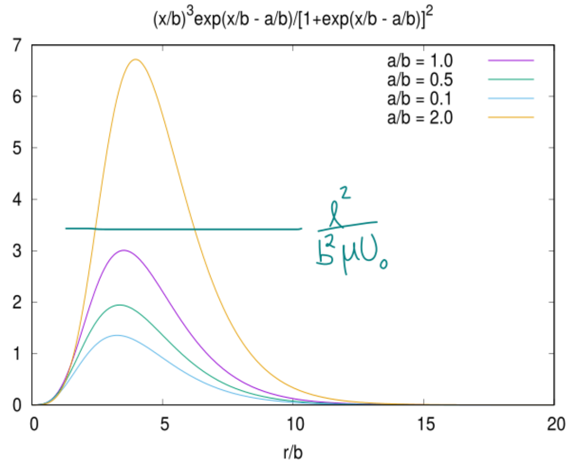
$$\frac{\partial V}{\partial r} = \frac{U_0 e^{\frac{r-a}{b}}}{b\left[1+e^{\frac{r-a}{b}}\right]^2} - \frac{\ell^2}{\mu r^3} = 0$$

$$\to \frac{U_0 e^{\frac{r-a}{b}} r^3}{b\left[1+e^{\frac{r-a}{b}}\right]^2} - \frac{\ell^2}{\mu} = 0$$

$$\to \boxed{\frac{e^{\frac{r-a}{b}}\left(\frac{r}{b}\right)^3}{\left\{1+e^{\frac{r-a}{b}}\right]^2} - \frac{\ell^2}{b^2\mu U_0} = 0}$$

óbein jafna fyrir r/b sem ákvarðar lágmark virka mættisins V(r), skoðum vinstri liðinn á grafi

(3)




Því sést að hverfiþunginn þarf að vera smár fyrir vist gildi á a/b til að tvær lausnir finnist. Þegar við athugum myndirnar á síðu 1 sést að önnur lausnin er stöðug, en hin er óstöðug. Þegar hverfiþunginn eykst kemur að því að þessir tveir punktar renna saman og fyrir enn hærri hverfiþunga er engin lausn til lengur fyrir stöðuga hringbraut. Lægri lausnin fyrir r/b er stöðug, en hin er óstöðug.

(3) Hve háan hverfiþunga getur ögn á hringbraut haft. Verðum að finna það fyrir gefin gildi á a/b. Athugum hámarkið á fallinu á þessari mynd

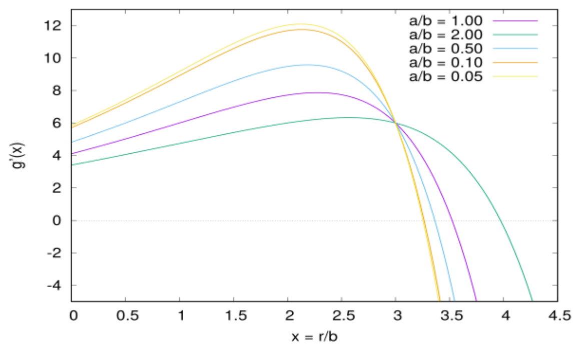
$$g(x) \equiv \frac{e^{x-\frac{a}{b}}x^3}{\left\{1+e^{x-\frac{b}{a}}\right]^2}, \quad x = \frac{r}{a}$$

(4)

$$g'(x) = \frac{e^{x-\frac{a}{b}}x^2}{\left\{1+e^{x-\frac{a}{b}}\right]^3}\left\{-2xe^{x-\frac{a}{b}} + x\left(1+e^{x-\frac{a}{b}}\right) + 3\left(1+e^{x-\frac{a}{b}}\right)\right\} = 0$$

$$\to e^{x-\frac{a}{b}}\left\{-2x+x+3\right\}+x+3=0$$

$$\to e^{x-\frac{a}{b}}\left[3-x\right]+x+3=0$$




Skoðum á grafi sjáum greinilega lausn fyrir hvert a/b gildi. Leitum að núllstöðvum með wxmaxima (find root) og berum saman við graf á næstu síðu

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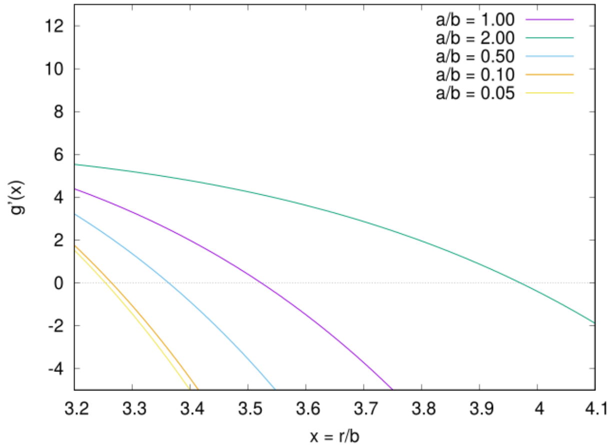


| $a/b$ | $x = \frac{r}{b}$ |
|---|---|
| 1 | 3,523 |
| 2 | 3,971 |
| 0,5 | 3,363 |
| 0,10 | 3,265 |
| 0,05 | 3,254 |

síðan notum við

$$\boxed{\ell^2 = g\left(\frac{r}{b};\frac{a}{b}\right) b^2 \mu U_0}$$

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**Dæmi 2** Braut $r = k\theta$

Er þannig braut möguleg í miðlægu mætti? Finna þá $F(r)$ og $U(r)$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$$\hookrightarrow \left\{\frac{2k^2}{(k\theta)^3} + \frac{1}{k\theta}\right\} = \frac{2k^2}{r^3} + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$$\rightarrow F(r) = -\frac{\ell^2}{\mu r^2}\left[\frac{2k^2}{r^3} + \frac{1}{r}\right] = -\frac{\ell^2 2k^2}{\mu r^5} - \frac{\ell^2}{\mu r^3}$$

$$F(r) = -\frac{\partial U}{\partial r} \rightarrow dU = -F(r)\,dr$$

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$$\rightarrow \int_{U(\infty)}^{U(r)} dU = -\int_{\infty}^{r} dr' F(r')$$

$$\rightarrow U(r) - U(\infty) = -\int_{\infty}^{r} dr'\left[-\frac{\ell^2 2k^2}{\mu r^5} - \frac{\ell^2}{\mu r^3}\right]$$

$$= \left[-\frac{\ell^2 k^2}{2\mu r^4} - \frac{\ell^2}{2\mu r^2}\right]\Bigg|_{\infty}^{r} = -\frac{\ell^2}{2\mu r^2}\left[1 + \frac{k^2}{r^2}\right]$$

Síðan er í þessu tilfelli hægt að setja mættisorkuna 0 í óendanlegu. Takið eftir að $U(r)$ er einhalla og krafturinn er aðdráttarkraftur

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**Dæmi 3** $r = k\tanh\theta$ ?

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$$\hookrightarrow \frac{d^2}{d\theta^2}\left(\frac{1}{k\tanh\theta}\right) + \frac{1}{k\tanh\theta} = \frac{1}{k}\left\{\frac{2\operatorname{sech}^2\theta}{\tanh\theta} + \frac{2\operatorname{sech}^4\theta}{\tanh^3\theta} + \frac{1}{\tanh\theta}\right\}$$

$$= -\frac{\mu r^2}{\ell^2}F(r)$$

þar sem

$$\operatorname{sech}\theta = \frac{1}{\cosh\theta}$$

$$\rightarrow F(r) = -\frac{\ell^2}{k\mu r^2}\left[\frac{2\operatorname{sech}^2\theta}{\tanh\theta} + \frac{2\operatorname{sech}^4\theta}{\tanh^3\theta} + \frac{1}{\tanh\theta}\right]$$

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Notum

$$\tanh\theta = \frac{r}{k}\,, \quad \operatorname{sech}^2\theta = 1-\tanh^2\theta$$

$$\rightarrow \boxed{F(r)} = -\frac{\ell^2}{k\mu r^2}\,\frac{k}{r}\left[2\left(1-\frac{r^2}{k^2}\right)+2\left(\frac{k}{r}\right)^2\left(1-\frac{r^2}{k^2}\right)^2+1\right]$$

$$= -\frac{\ell^2 k}{k\mu r^3}\left\{3-2\left(\frac{r}{k}\right)^2+2\left(\frac{k}{r}\right)^2\left(1-2\left(\frac{r}{k}\right)^2+\left(\frac{r}{k}\right)^4\right)\right]$$

$$= -\frac{\ell^2}{\mu r^3}\left\{3-4-2\left(\frac{r}{k}\right)^2+2\left(\frac{r}{k}\right)^2+2\left(\frac{k}{r}\right)^2\right]$$

$$= -\frac{\ell^2}{\mu r^3}\left\{2\left(\frac{k}{r}\right)^2-1\right] = \boxed{-\frac{2\ell^2 k^2}{\mu r^5}+\frac{\ell^2}{\mu r^3}}$$

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$$F(r) = -\frac{\partial U}{\partial r}$$

$$\rightarrow dU = -F(r)\,dr$$

$$\int_{U(\infty)}^{U(r)} dU = -\int_{\infty}^{r} dr'\,F(r')$$

$$\rightarrow U(r)-U(\infty) = -\int_{\infty}^{r} dr'\left\{-\frac{2(\ell k)^2}{\mu r^5}+\frac{\ell^2}{\mu r^3}\right]$$

$$= \left\{-\frac{\ell^2 k^2}{2\mu r^4}+\frac{\ell^2}{2\mu r^2}\right]\Bigg|_{\infty}^{r}$$

$$= -\frac{\ell^2 k^2}{2\mu r^4}+\frac{\ell^2}{2\mu r^2}$$

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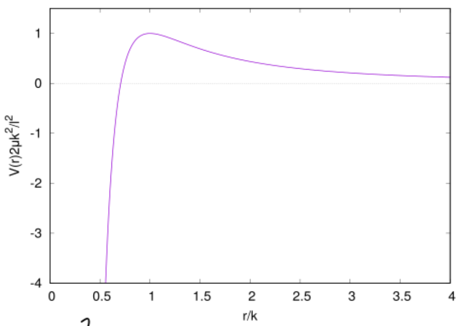
Getum sett $U(\infty) = 0$

$$\rightarrow \boxed{U(r) = \frac{\ell^2}{2\mu r^2}\left\{1-\frac{k^2}{r^2}\right]}$$




Skoðum virka mættið

$$V(r) = U(r)+\frac{\ell^2}{2\mu\left(\frac{r}{k}\right)^2 k^2} = \frac{\ell^2}{2\mu k^2\left(\frac{r}{k}\right)^2}\left\{2-\left(\frac{k}{r}\right)^2\right]$$

því lítur út fyrir að brautin sé aðeins fyrir ögn sem er bundin í mættinu. Hnitið r/k er takmarkað í brautarhreyfingunni

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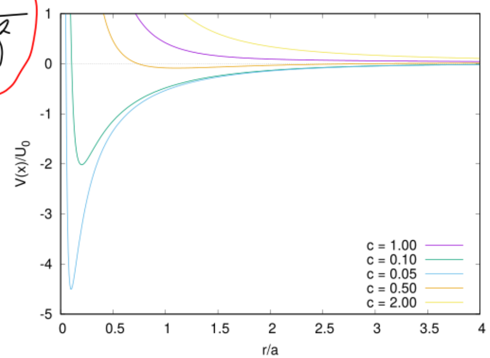
Dæmi 4

$$U(r) = -U_0\,\frac{e^{-r/a}}{1-e^{-r/a}}\,, \quad a>0$$

① Virka mættið $V(r) = U(r)+\dfrac{\ell^2}{2\mu r^2}$, $\ell = \mu r^2\dot{\theta}$ fasti

$$\boxed{\frac{V(r)}{U_0} = -\frac{e^{-r/a}}{1-e^{-r/a}}+\frac{\ell^2}{2\mu U_0 a^2\left(\frac{r}{a}\right)^2}}$$




Fyrir grafið setjum

$$c = \frac{\ell^2}{2\mu U_0 a^2} \leftarrow$$

víddarlaus fasti

(2) Lágmarkið í $V(r)$ bendir til þess að fyrir nógu lágan hverfiþunga séu til hringbrautir. Stöðugleikinn kemur í ljós í næstu liðum.

(3) Hver er mesti hverfiþungi sem ögn á hringhreyfingu getur haft í mættinu?

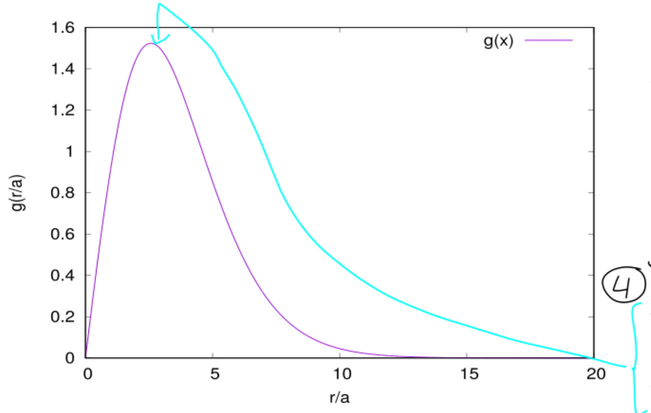
$$\frac{\partial V}{\partial r} = \frac{U_0 e^{-r/a}}{a\left(1-e^{-r/a}\right)^2}\left[\left(1-e^{-r/a}\right)+1\right] - \frac{\ell^2}{\mu r^3} = 0$$

$$\rightarrow \boxed{\left(\frac{r}{a}\right)^3 \frac{e^{-\frac{2r}{a}}}{\left(1-e^{-r/a}\right)^2} - \frac{\ell^2}{\mu U_0 a^2} = 0}$$

Óbein jafna til að ákvarða geisla hringbrautar í $V(r)$, skoðum hana betur á næstu síðu



Setjum $g(x) = x^3 \dfrac{e^{-2x}}{\left(1-e^{-x}\right)^2}$ og setjum á graf




Því eru til fyrir nógu lágan hverfiþunga tvær hringbrautir, sú innri stöðug og sú ytri óstöðug. Það er erfitt að sjá óstöðuga möguleikann frá grafinu af $V(r)$ hér að framan

(4) Finnum mesta hverfiþungann sem ögn á hringbraut getur haft

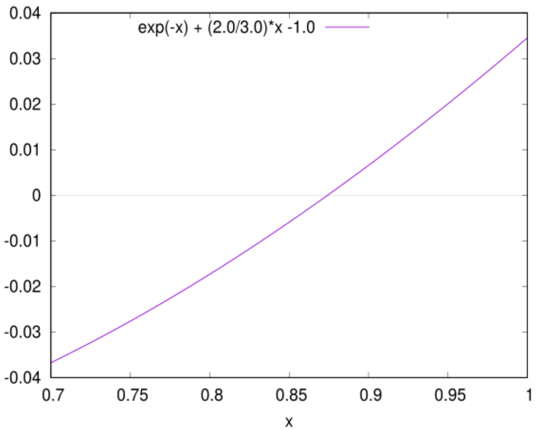
$$g'(x) = \frac{x^2 e^{-2x}}{\left(1-e^{-x}\right)^3}\left\{-2xe^{-x} - 2x\left(1-e^{-x}\right) + 3\left(1-e^{-x}\right)\right\} = 0$$



ef $x \neq 0$

$$\rightarrow \boxed{e^{-x} + \frac{2}{3}x - 1 = 0}$$

óbein jafna til að finna hámark $g(x)$, skoðum graf




ein lausn sem finna má með wxmaxima: $x = 0{,}8742$

$$\rightarrow \boxed{\ell^2 = \mu U_0 a^2 g(0{,}8742)}$$



(5) Smáar sveiflur um hringbraut

$$L = \frac{\mu \dot{r}^2}{2} + \frac{\mu (r\dot\theta)^2}{2} + U_0 \frac{e^{-r/a}}{1-e^{-r/a}}$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = 0 \rightarrow \mu r\dot\theta^2 + \frac{U_0}{a}\frac{e^{-2r/a}}{\left(1-e^{-r/a}\right)^2} - \mu\ddot r = 0$$

og $\ell = \mu r^2\dot\theta$ fasti

$$\rightarrow \boxed{\ddot r - \frac{\ell^2}{\mu^2 r^3} + \frac{U_0}{\mu a}\frac{e^{-\frac{2r}{a}}}{\left(1-e^{-r/a}\right)^2} = 0}$$

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Hringbraut -> $\dot{r}, \ddot{r} = 0$

línuleg nálgun um jafnvægispunkt

$$r = r_0 + \delta, \quad \dot{r} = \dot{\delta}, \quad \ddot{r} = \ddot{\delta}$$

$$\rightarrow \frac{1}{(r_0+\delta)^3} \simeq \frac{1}{r_0^3}\left[1 - 3\frac{\delta}{r_0} - - -\right]$$

$$e^{-\frac{2r}{a}} = e^{-\frac{2}{a}(r_0+\delta)} = e^{-\frac{2r_0}{a}\left(1+\frac{\delta}{r_0}\right)} \simeq e^{-\frac{2r_0}{a}}\left(1+\frac{\delta}{r_0}+\cdots\right)$$

$$U(r) \rightarrow \frac{U_0}{a}\frac{e^{-\frac{2r_0}{a}}}{(1-e^{-r_0/a})^2}\left\{1+\frac{\delta}{r_0}+\cdots\right]$$

$$\rightarrow \ddot{\delta} - \frac{\ell^2}{\mu^2 r_0^3}\left\{1 - \frac{3\delta}{r_0}\right] + \frac{U_0}{\mu a}\frac{e^{-\frac{2r_0}{a}}}{(1-e^{-r_0/a})^2}\left\{1+\frac{\delta}{r_0}\right] \simeq 0$$

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sem gefur

$$\ddot{\delta} + \left\{\frac{3\ell^2}{\mu^2 r_0^4} + \frac{U_0}{\mu a r_0}\frac{e^{-\frac{2r_0}{a}}}{(1-e^{-r_0/a})^2}\right]\delta = 0$$

og jafnvægisskilyrðin aftur

$$\frac{\ell^2}{\mu^2 r_0^3} = \frac{U_0}{\mu a}\frac{e^{-\frac{2r_0}{a}}}{(1-e^{-r_0/a})^2}$$

$$\rightarrow \Omega^2 = \frac{3\ell^2}{\mu^2 r_0^4} + \frac{U_0\, e^{-\frac{2r_0}{a}}}{a r_0 \mu (1-e^{-r_0/a})^2}$$

$$= \frac{4\ell^2}{\mu^2 r_0^4} \qquad \rightarrow \boxed{\Omega = \frac{2|\ell|}{\mu r_0^2}}$$