1

Dæmi 1

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}F(q,t)$$

Höfum sýnt að
$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

L' er fall af sömu breytum og L

$$\rightarrow \frac{\partial L'}{\partial q} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) = 0$$ ætti að gilda

Sannreynum

$$\frac{\partial L'}{\partial q} = \frac{\partial L}{\partial q} + \frac{\partial}{\partial q}\left(\frac{dF}{dt}\right)$$

$$\frac{\partial L'}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}}\left(\frac{dF}{dt}\right) \rightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}}\left(\frac{dF}{dt}\right)\right\} \qquad (*)$$

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Athugum heildarafleiðuna af F

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t} \rightarrow \frac{\partial}{\partial \dot{q}}\left(\frac{dF}{dt}\right) = \frac{\partial F}{\partial q}$$

$F = F(q,t)$

notum í (*)

$$\rightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left(\frac{\partial F}{\partial q}\right)$$

$$\rightarrow \frac{\partial L'}{\partial q} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} + \frac{\partial}{\partial q}\left(\frac{dF}{dt}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial q}\right) = 0$$

$$= \underbrace{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)}_{\text{E-L fyrir } L} + \underbrace{\left[\frac{\partial}{\partial q}\left(\frac{dF}{dt}\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial q}\right)\right]}_{=0} = 0$$

3

Dæmi 2

Ögn með massa m hreyfist í kúluyfirborði án þess að ytri kraftar verki á hana. (Heimur hennar er kúluyfirborð með geisla R)

Nýtum Ex. 6.3 í DC, og munum að R er fasti

① 2 alhnit -- yfirborð, hornin úr kúluhnitum spanna það

② $$\rightarrow L = \frac{mR^2}{2}\left[\dot{\theta}^2 + (\sin^2\theta)\dot{\phi}^2\right] = L(\theta, \dot{\phi}, \dot{\theta})$$

③ Alskriðþungar

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}$$ ekki fasti, því θ kemur fyrir í L

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2(\sin^2\theta)\dot{\phi}$$ fasti, því ϕ er rásuð breyta

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Fall Hamiltons

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L = \frac{mR^2}{2}\left[\dot{\theta}^2 + (\sin^2\theta)\dot{\phi}^2\right]$$
$$= \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{2mR^2\sin^2\theta} = H(p_\theta, p_\phi, \theta)$$

í raun sést hér að H og L eru bæði einfaldlega T í mismunandi breytum

④ Hreyfijöfnur Hamiltons

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{mR^2}\frac{p_\phi^2\cos\theta}{\sin^3\theta}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mR^2\sin^2\theta}$$

Þetta 1. sigs hreyfijöfnuhneppi er mjög heppilegt beint í tölulega reikninga. Hægt er að sýna að hreyfing agnarinnar er alltaf eftir stórhring, en ég sleppti að biðja um það hér.

(5)

Ögnin finnur fyrir engu ytra mætti. Fall Hamiltons er einungis hreyfiorkan. Enginn núningskraftur vinnur á móti hreyfingunni, Fall Hamiltons er ekki háð tíma, þess vegna er fall Hamiltons hér heildarorka agnarinnar.

Dæmi 3

Ögn með massa m hreyfist í yfirborði sem myndast þegar fleygboga er snúið um z-ás kartísks hnitakerfis. Yfirborðið er kyrrt í þyngdarsviði. Notið sívalningshnitin r (mælt frá samhverfuás yfirborðsins z-ásnum) og φ sem alhnit

Notum sívalningshnit r, ϕ

$$\to \quad L = \frac{m}{2}\left[\dot r^2 + r^2\dot\phi^2 + \dot z^2\right] - mgz$$

Fleygbogayfirborð $\quad az = r^2 = x^2 + y^2$

$$z = \frac{r^2}{a} \to \dot z = \frac{2r\dot r}{a}$$

$$\to \quad \boxed{L = \frac{m}{2}\left\{\dot r^2 + r^2\dot\phi^2 + \frac{4r^2}{a^2}\dot r^2\right\} - mg\frac{r^2}{a}}$$

$$= \frac{m}{2}\left\{\dot r^2\left(1 + \frac{4r^2}{a^2}\right) + (r\dot\phi)^2\right\} - mg\frac{r^2}{a} = L(r, \dot r, \phi)$$

(2) Alskriðþungar og H

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = mr^2\dot\phi = \text{fasti}$$ því φ er rásuð breyta

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r\left[1 + \frac{4r^2}{a^2}\right]$$

$$H = p_\phi\dot\phi + p_r\dot r - L = mr^2\dot\phi^2 - \frac{m}{2}r^2\dot\phi^2 + m\dot r^2\left[1 + \frac{4r^2}{a^2}\right] - \frac{m}{2}\dot r^2\left[1 + \frac{4r^2}{a^2}\right] + mg\frac{r^2}{a}$$

$$= \frac{p_r^2}{2m\left[1 + \frac{4r^2}{a^2}\right]} + \frac{p_\phi^2}{2mr^2} + mg\frac{r^2}{a}$$

(3) Hreyfijöfnur Hamiltons

$$\dot r = \frac{\partial H}{\partial p_r} = \frac{p_r}{m\left\{1 + \frac{4r^2}{a^2}\right\}}$$

$$\dot p_r = -\frac{\partial H}{\partial r} = -2mg\frac{r}{a} + \frac{p_\phi^2}{mr^3} + \frac{p_r^2\,\frac{8r}{a^2}}{2m\left\{1 + \frac{4r^2}{a^2}\right\}^2}$$

$$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2}$$

$$\dot p_\phi = -\frac{\partial H}{\partial \phi} = 0$$

φ er rásuð breyta

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④ $$L = \frac{m}{2}\left\{\dot r^2\left(1+\frac{4r^2}{a^2}\right) + r^2\dot\phi^2\right\} - mg\frac{r^2}{a}$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = 0$$

$$\hookrightarrow \frac{m}{2}\dot r^2\frac{8r}{a^2} + mr\dot\phi^2 - 2mg\frac{r}{a} - \frac{d}{dt}\left[m\dot r\left(1+\frac{4r^2}{a^2}\right)\right] = 0$$

$$\to \dot r^2\frac{4r}{a^2} + r\dot\phi^2 - 2g\frac{r}{a} - \ddot r\left(1+\frac{4r^2}{a^2}\right) - \frac{\dot r\, 8r\dot r}{a^2} = 0$$

$$\to \boxed{\ddot r\left(1+\frac{4r^2}{a^2}\right) + \frac{4r\dot r^2}{a^2} - r\dot\phi^2 + 2g\frac{r}{a} = 0}$$

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⑤ Finnið tíðni smárra sveiflna agnarinnar þegar $d\phi/dt = 0$

Hér væri hægt að nota hreyfijöfnuna í síðasta lið, en ég stytti mér leið

$$\frac{d\phi}{dt} = 0 \to P_\phi = 0 \to H = \frac{P_r^2}{2m\left\{1+\frac{4r^2}{a^2}\right\}} + \frac{mg}{a}r^2$$

Við þekkjum fall Hamiltons fyrir hreintóna sveifil

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2$$

Athugun á okkar H sýnir að það lýsi línulegum sveifli ef $\frac{r^2}{a^2} \ll 1$

$$\to H \to \simeq \frac{P_r^2}{2m} + \frac{mg}{a}r^2$$

$$\to \frac{mg}{a^2} = \frac{1}{2}m\omega^2 \to \boxed{\omega = \sqrt{\frac{2g}{a}}}$$

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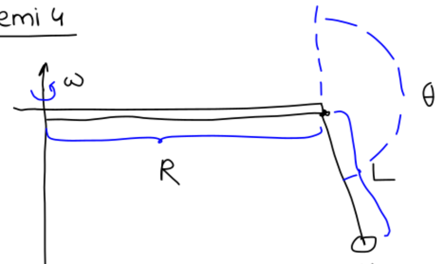
Dæmi 4



ég mæli hornið svona til að nýta venjuleg kúluhnit

$$x = R\cos(\omega t) + L\sin\theta\cos\phi$$

$$y = R\sin(\omega t) + L\sin\theta\sin\phi$$

$$z = L\cos\theta$$

$$U = mgL\cos\theta$$

$$\dot x = -\omega R\sin(\omega t) + L\dot\theta\cos\theta\cos\phi - L\dot\phi\sin\theta\sin\phi$$

$$\dot y = \omega R\cos(\omega t) + L\dot\theta\cos\theta\sin\phi + L\dot\phi\sin\theta\cos\phi$$

$$\dot z = -\dot\theta L\sin\theta$$

$$T = \frac{m}{2}\left\{\dot x^2 + \dot y^2 + \dot z^2\right\}$$

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$$T = \frac{m}{2}\Big[L^2\dot\theta^2 + (L\sin\theta\cdot\dot\phi)^2 + \omega^2R^2\left[\sin^2(\omega t) + \cos^2(\omega t)\right]$$
$$+ 2\omega RL\left(-\sin(\omega t)\cos\theta\cos\phi\cdot\dot\theta + \sin(\omega t)\sin\theta\sin\phi\cdot\dot\phi\right)$$
$$+ 2\omega RL\left(\cos(\omega t)\cos\theta\sin\phi\cdot\dot\theta + \cos(\omega t)\sin\theta\cos\phi\cdot\dot\phi\right)\Big]$$

$$\to \boxed{L = \frac{m}{2}\Big\{(L\dot\theta)^2 + (L\sin\theta\cdot\dot\phi)^2 + (\omega R)^2 + \dot\theta\, 2\omega RL\cos\theta\sin(\phi-\omega t) + \dot\phi\, 2\omega RL\sin\theta\cos(\phi-\omega t)\Big\} - mgL\cos\theta}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 0$$

$$\rightarrow mL^2 \sin\theta\cos\theta\cdot\dot\phi^2 + mgL\sin\theta - \dot\theta\, 2\omega RL\sin\theta\sin(\phi-\omega t)$$

$$+ \dot\phi\, 2\omega RL\cos\theta\cos(\phi-\omega t)$$

$$- \frac{d}{dt}\left\{mL^2\dot\theta + 2\omega RL\cos\theta\sin(\phi-\omega t)\right\} = 0$$

$$= \dot\phi^2 mL^2\sin\theta\cos\theta + mgL\sin\theta - \cancel{\dot\theta\, 2\omega RL\sin\theta\sin(\phi-\omega t)}$$

$$+ \cancel{\dot\phi\, 2\omega RL\cos\theta\cos(\phi-\omega t)}$$

$$- mL^2\ddot\theta + \cancel{2\omega RL\dot\theta\sin\theta\sin(\phi-\omega t)}$$

$$- 2\omega RL\cos\theta\cos(\phi-\omega t)\cdot(\cancel{\dot\phi}-\omega)$$

$$\rightarrow \boxed{\ddot\theta - \dot\phi^2\sin\theta\cos\theta - 2\omega^2 RL\cos\theta\cos(\phi-\omega t) - \frac{g}{L}\sin\theta = 0}$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = 0$$

$\hookrightarrow \ldots$

$$\boxed{\ddot\phi + 2\cot\theta\cdot\dot\theta\dot\phi + \frac{2\omega^2 R}{mL}\,\frac{\sin(\phi-\omega t)}{\sin\theta} = 0}$$

Mér sýnist samanburður við "8.6 Example" í bók DC sýna að þetta verði hreyfijöfnurnar þegar $\omega = 0$, en það þarf að beita einni tímaafleiðu til að sjá það fyrir þessa seinni jöfnu. Það er athyglisvert að snúningur stangarinnar kemur eins og þvingunarliður. Því er öruggt að H lýsir ekki heildarorku kerfisins, þetta er í raun opið kerfi, þar sem snúningurinn bætir við og tekur út orku