1

Dæmi 1 Massamiðja hálfkúluhvels með innri og ytri geisla a og b, og fastan þéttleika ρ á rúmmál

Setjum botnflöt hvelsins í x-y-sléttu. Samhverfa gefur $x_{cm}=0$ og $y_{cm}=0$

$$V = \underbrace{\int_0^{\pi/2} d\theta \sin\theta}_{=1} \int_0^{2\pi} d\phi \int_a^b r^2 dr$$

Kúluhnit falla best að verkefninu

$$= 2\pi \int_a^b r^2 dr = 2\pi \left[\frac{r^3}{3} \Big|_a^b \right]$$

$z_{cm} \neq 0$

$$= \frac{2\pi}{3} \left\{ b^3 - a^3 \right\}$$

Rúmmál hálfkúluhvelsins

2

$$\boxed{z_{cm}} = \frac{1}{V} \int_0^{\pi/2} d\theta \sin\theta \int_0^{2\pi} d\phi \int_a^b r^2 dr \, \underbrace{z}_{r\cos\theta = z}$$

$$= \frac{1}{V} \int_0^{\pi/2} d\theta \sin\theta \cos\theta \int_0^{2\pi} d\phi \int_a^b r^3 dr$$

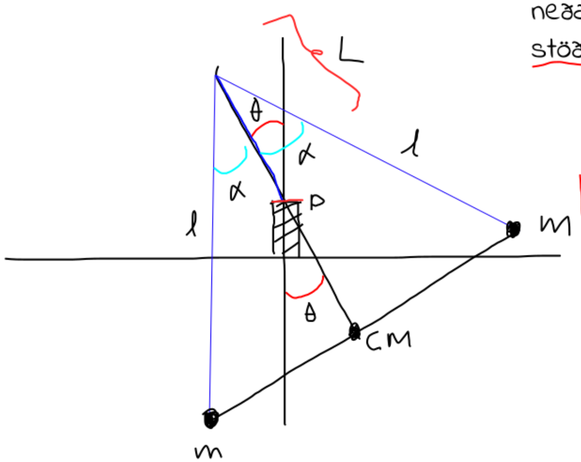
$$= \frac{2\pi}{V} \underbrace{\int_0^{\pi/2} d\theta \sin\theta \cos\theta}_{1/2} \left[\frac{r^4}{4} \Big|_a^b \right]$$

$$V = \frac{2\pi}{3}\left[b^3 - a^3\right]$$

$$= \frac{2\pi}{4V} \left\{ b^4 - a^4 \right\} \frac{1}{2} = \frac{\pi}{4V} \left\{ b^4 - a^4 \right\} = \boxed{\frac{3}{8} \left[\frac{b^4 - a^4}{b^3 - a^3} \right]}$$

3

Dæmi 2 Skoðum aðeins sveiflur í sléttu blaðsins, hinar eru eins

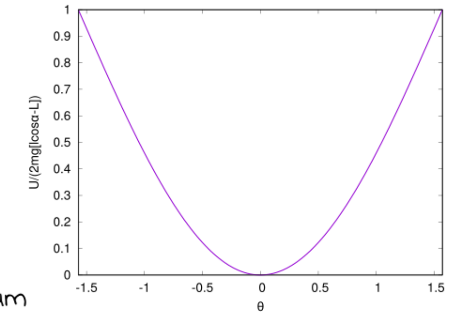


Ég vel $\boxed{l\cos\alpha > L}$

því þá lendir massamiðjan fyrir neðan P og kerfið getur verið í stöðugu jafnvægi

Stöðuorkan er þá

$$\boxed{U = g2m(l\cos\alpha - L)(1-\cos\theta)}$$



Um hornið gildir $-\pi/2 \leq \theta \leq \pi/2$

Fyrir stærra horn dettur stubburinn af stallinum

4

$$T = 2\frac{m}{2}\left[(l\cos\alpha - L)\dot{\theta}\right]^2, \quad U = g2m(l\cos\alpha - L)(1-\cos\theta)$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \longrightarrow -2m(l\cos\alpha - L) g \sin\theta - (l\cos\alpha - L)^2 2m\ddot{\theta} = 0$$

eða

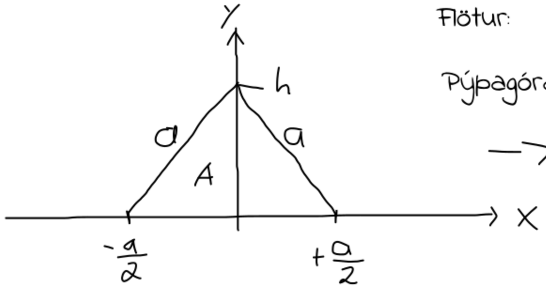
$$\boxed{\ddot{\theta} + \frac{g}{(l\cos\alpha - L)} \theta \simeq 0}$$

í línulegri nálgun

Kerfið hagar sér eins og pendúll, og fyrir smáar sveiflur fæst

$$\boxed{\omega = \sqrt{\frac{g}{(l\cos\alpha - L)}}}$$

Dæmi 3 Finna massamiðju jafnhliða þríhyrnings



Flötur: $A = \left(\frac{a}{2}h\right)$

Pýþagóras: $h^2 + \left(\frac{a}{2}\right)^2 = a^2$

$\longrightarrow \quad h = \frac{\sqrt{3}a}{2}$

$A = \frac{\sqrt{3}a^2}{4}$

Lína í gegnum (a/2,0) og $\left(0, \frac{\sqrt{3}a}{2}\right)$ $\qquad x_{cm} = 0$

$$y = -\sqrt{3}x + \frac{\sqrt{3}a}{2}$$

$$\longrightarrow \boxed{y_{cm} = \frac{2\sigma \int_0^{a/2} dx \int_0^{-\sqrt{3}x + \frac{\sqrt{3}a}{2}} dy\, y}{\frac{\sqrt{3}a^2}{4}} = \frac{4}{\sqrt{3}a^2}\int_0^{a/2} dx\, \frac{\left(-\sqrt{3}x + \frac{\sqrt{3}a}{2}\right)^2}{2}}$$

$$= \frac{2}{\sqrt{3}}\frac{2}{a^2}\int_0^{a/2} dx \left\{3x^2 - 3xa + \frac{3}{4}a^2\right\}$$

$$= \frac{4}{\sqrt{3}a^2}\left[\left(x^3 - \frac{3a}{2}x^2 + \frac{3a^2}{4}x\right)\Big|_0^{a/2}\right]$$

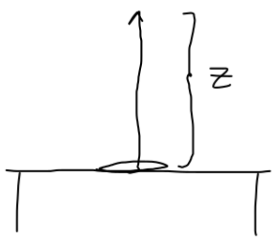
$$= \frac{4}{\sqrt{3}a^2}\left\{\left(\frac{a}{2}\right)^3 - \frac{3a}{2}\left(\frac{a}{2}\right)^2 + \frac{3a^2}{4}\left(\frac{a}{2}\right)\right\}$$

$$= \frac{4a^3}{\sqrt{3}a^2}\left[\frac{1}{8} - \frac{3}{8} + \frac{3}{8}\right] = \frac{4a^3}{\sqrt{3}a^2 8} = \frac{a}{2\sqrt{3}}$$

$$\boxed{= \frac{\sqrt{3}a}{6} = y_{cm}}$$

Dæmi 4 Þjált reipi togað upp af borði með fastri hröðun a, finna kraftinn á höndina

Massi á lengd: λ, lengd reipis L



Þyngd reipis á lofti: $\lambda z g$

Atlag á hönd: $F_{imp} = \frac{d}{dt}\left[(\lambda z)v\right]$

$$v = at, \quad v(0) = 0$$

$$\ddot{z} = \frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = \frac{dv}{dz}v = a$$

$$\longrightarrow \quad v^2 = 2az \quad \text{ef} \quad v(0) = 0$$

$$F_{imp} = \frac{d}{dt}\left\{(\lambda z)v\right\} = \left(\lambda\frac{dz}{dt}\right)v + \lambda z\frac{dv}{dt}$$

$$= \lambda v^2 + \lambda z a = \lambda 2az + \lambda z a$$

$$= 3\lambda a z$$

Því fæst fyrir heildarkraftinn

$$\boxed{F(z) = \lambda z g + 3\lambda a z = \lambda z\left[g + 3a\right]}$$

atlag

Þar sem við höfum ekki tiltekið stefnu kraftsins