1

Dæmi 1 Ögn í mættinu $U(r) = -U_0 \exp\left\{-\left(\frac{r}{a}\right)^2\right\}$

Virkamættið er $V(r) = U(r) + \frac{\ell^2}{2\mu r^2}$, $\ell = \mu r^2 \dot\theta$ fasti

Finnum lágmark í V(r)

$$\frac{\partial V}{\partial r} = \frac{U_0}{a}\frac{2r}{a}\exp\left\{-\left(\frac{r}{a}\right)^2\right\} - \frac{\ell^2}{\mu r^3} = 0$$

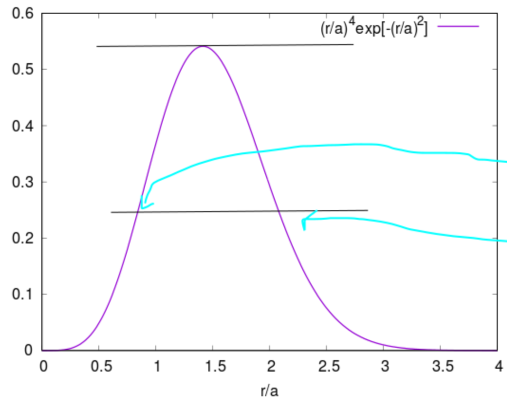
$$\rightarrow \boxed{\exp\left\{-\left(\frac{r}{a}\right)^2\right\} = \frac{a^2\ell^2}{2\mu U_0 r^4} = \frac{\ell^2}{2U_0\mu a^2}\left(\frac{a}{r}\right)^4}$$

Óbein jafna fyrir útgildi sem getur verið lágmark eða hámark, umskrifum sem

$$\boxed{\left(\frac{r}{a}\right)^4 \exp\left\{-\left(\frac{r}{a}\right)^2\right\} = \underbrace{\frac{\ell^2}{2U_0\mu a^2}}_{\text{fasti}}}$$

Þessi útsetning nýtist enn betur á grafi, skoðum

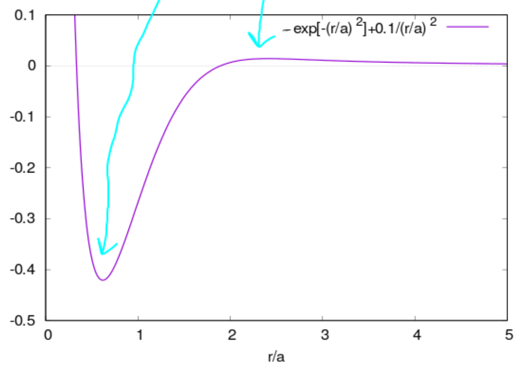
2



Því þarf fastinn að vera nógu lágur til að bjóða upp á lausn á óbeinu jöfnunni. Þegar tvær lausnir eru, er aðeins sú fyrri fyrir lágmark, en sú seinni fyrir staðbundið hámark

Við gætum fundið gildi fyrir r þegar mættið tekur lágmark, við köllum það einfaldlega r_0

Eins getum við fundið efri mörkin fyrir herfiþungann l, en mikilvægast er að sjá að til eru efri mörk á honum fyrir hringbraut



2b

Finnum hvar fallið g(x) tekur hámark

$$g(x) = x^4 e^{-x^2} \rightarrow g'(x) = 4e^{-x^2}\left(1 - \frac{x^2}{2}\right) = 0$$

gefur $x^2 = 2 \rightarrow x = \frac{r}{a} = \sqrt{2}$

Í þessum punkti er

$$4e^{-2} = \frac{\ell_{max}^2}{2U_0\mu a^2} \rightarrow \boxed{\ell_{max}^2 = 8U_0\mu a^2 e^{-2}}$$

Gildi virka mættisins í þessum punkti

$$V(\sqrt{2}a) = -U_0 e^{-2} + \frac{\ell_{max}^2}{2\mu\sqrt{2}^2 a^2} = -U_0 e^{-2} + \frac{8U_0\mu a^2 e^{-2}}{2\mu 2 a^2}$$

$$= -U_0 e^{-2} + 2e^{-2}U_0 \; \underline{= U_0 e^{-2}}$$

3

Skoðum aðeins skilyrðið fyrir hringlaga braut

$$L = \frac{\mu\dot r^2}{2} + \frac{\mu(r\dot\theta)^2}{2} + U_0\exp\left\{-\left(\frac{r}{a}\right)^2\right\}$$

Euler-Lagrange $\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = 0$

$$\rightarrow \mu r\dot\theta^2 - \frac{U_0 2}{a}\left(\frac{r}{a}\right)\exp\left[-\left(\frac{r}{a}\right)^2\right] - \mu\ddot r = 0$$

Notum $\ell = \mu r^2\dot\theta$ og umskrifum hreyfijöfnuna

$$\ddot r - \frac{\ell^2}{\mu^2 r^3} + \frac{U_0 2}{\mu a}\left(\frac{r}{a}\right)\exp\left\{-\left(\frac{r}{a}\right)^2\right\} = 0$$

Hringbraut $\underline{\dot r, \ddot r = 0}$

$$\rightarrow \exp\left\{-\left(\frac{r}{a}\right)^2\right\} = \frac{\ell^2}{2U_0\mu a^2}\left(\frac{a}{r}\right)^4$$

sama skilyrði og áður!

Dæmi 2   Lennard-Jones mætti atóma

$$U(r) = E_0\left\{\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right\}$$

Miðlægt mætti

$$\rightarrow\quad L = \underbrace{\frac{M}{2}|\dot{\vec R}|^2}_{L_{cm}} + \overbrace{\frac{\mu}{2}|\dot{\vec r}|^2 - U(r)}^{L_{rel}}$$

$$L_{rel} = \frac{\mu}{2}\left\{\dot r^2 + (r\dot\theta)^2\right\} - E_0\left\{\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right\}$$

$$\frac{\partial L_{rel}}{\partial \theta} = 0,$$ θ er rásuð breyta (e. cyclic) $\mu r^2\dot\theta = \ell$ fasti

Virka mættið

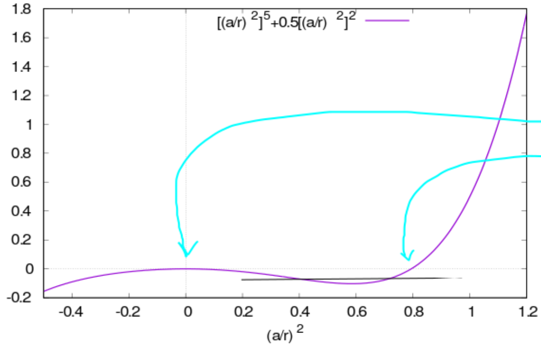
$$\boxed{V(r) = E_0\left\{\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right\} + \frac{\ell^2}{2\mu r^2}}$$

Finnum jafnvægisfjarlægð í kerfinu

$$\frac{\partial}{\partial r}V(r) = E_0\left\{-\frac{12}{a}\left(\frac{a}{r}\right)^{13} + \frac{6}{a}\left(\frac{a}{r}\right)^{7}\right\} - \frac{\ell^2}{\mu r^3} = 0$$

Umskrifum sem

$$\boxed{\left[\left(\frac{a}{r}\right)^2\right]^5 - \frac{1}{2}\left[\left(\frac{a}{r}\right)^2\right]^2 + \frac{\ell^2}{12E_0\mu a^2} = 0} \quad (*)$$



Jafnvægislengdin r_0 er rauntala því er lausn aðeins möguleg á bilinu $0<(a/r)^2<0.8$

Nú gefur rótin til hægri jafnvægisfjarlægðina, en hin staðbundið hámark mættisins

Smáar sveiflur. Leiðum út hreyfijöfnuna með Euler-Lagrange

$$-\mu\ddot r + \mu r\dot\theta^2 - E_0\left\{-\frac{12}{a}\left(\frac{a}{r}\right)^{13} + \frac{6}{a}\left(\frac{a}{r}\right)^{7}\right\} = 0$$

Notum $\mu r^2\dot\theta = \ell$

$$\rightarrow\quad \boxed{\ddot r - \frac{\ell^2}{\mu^2 r^3} - \frac{E_0}{\mu}\left\{-\frac{12}{a}\left(\frac{a}{r}\right)^{13} + \frac{6}{a}\left(\frac{a}{r}\right)^{7}\right\} = 0}$$

Athugun á stöðugri hringbraut gefur sama skilyrði og hér á undan, en smáar sveiflur, jafnvægislausn r_0 Smátt frávik δ

$$r = r_0 + \delta, \quad \dot r = \dot r_0 + \dot\delta = \dot\delta, \quad \ddot r = \ddot\delta$$

$$\frac{1}{r^3} = \frac{1}{(r_0+\delta)^3} \simeq \frac{1}{r_0^3} - \frac{3\delta}{r_0^4} + \dots = \frac{1}{r_0^3}\left\{1 - 3\frac{\delta}{r_0}\dots\right.$$

$$\frac{1}{(r_0+\delta)^n} \simeq \frac{1}{r_0^n}\left\{1 - n\frac{\delta}{r_0}\dots\right.$$

Setjum inn í hreyfijöfnuna og höldum öllum fyrstastigs liðum

$$\ddot\delta - \frac{\ell^2}{\mu^2 r_0^3}\left[1 - 3\frac{\delta}{r_0}\right] + \frac{12E_0a^{12}}{\mu r_0^{13}}\left[1 - 13\frac{\delta}{r_0}\right] - \frac{6E_0a^6}{\mu r_0^7}\left[1 - 7\frac{\delta}{r_0}\right] = 0$$

Þessir þrír liðir eru jafnvægisskilyrðið og falla því út

Eftir stendur

$$\boxed{\ddot\delta + \underbrace{\left\{\frac{3\ell^2}{\mu^2 r_0^4} - \frac{156E_0a^{12}}{\mu r_0^{14}} + \frac{42E_0a^6}{\mu r_0^8}\right\}}\delta \simeq 0}$$

Hér er ω^2, grunntíðni smáu sveiflunnar. Þessa grunntíðni má einfalda með jafnvægisskilyrðinu og þannig losna við ℓ^2

$$\omega^2 = \frac{3}{r_0^4}\left[\frac{l^2}{\mu^2} - \frac{52E_0a^2}{\mu}\left(\frac{a}{r_0}\right)^{10} + \frac{14E_0a^2}{\mu}\left(\frac{a}{r_0}\right)^4\right]$$

Munum að lágmarksskilyrðin gáfu

$$\frac{l^2}{\mu^2} = -\frac{12E_0a^2}{\mu}\left(\frac{a}{r_0}\right)^{10} + \frac{6E_0a^2}{\mu}\left(\frac{a}{r_0}\right)^4$$

því fæst

$$\boxed{\omega^2 = \frac{3}{r_0^4}\left[-\frac{64E_0a^2}{\mu}\left(\frac{a}{r_0}\right)^{12} + \frac{20E_0a^2}{\mu}\left(\frac{a}{r_0}\right)^4\right]}$$

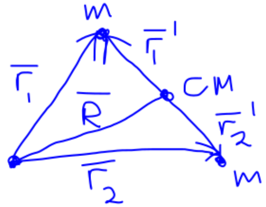
Dæmi 3 Tveir massar m á láréttu hálu borði víxlverkast með F=-kr

$$\boxed{L = \frac{M}{2}|\dot{\bar{R}}|^2 + \frac{\mu}{2}|\dot{\bar{r}}|^2 - U(r)}, \quad U(r) = \frac{k}{2}r^2$$

$$M = 2m, \quad \mu = \frac{m}{2}$$

$$\frac{\partial L}{\partial \bar{R}} = 0 \to \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{R}}}\right) = \frac{d\bar{P}_{cm}}{dt} = 0 \to \bar{P}_{cm} = M\dot{\bar{R}} \text{ fasti}$$

$$\boxed{L_{rel} = \frac{\mu}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2\right\} - \frac{k}{2}r}, \quad \bar{r} = \bar{r}_1' - \bar{r}_2'$$



$$\frac{\partial L_{rel}}{\partial \theta} = 0,$$ θ er rásuð breyta

$$\frac{d}{dt}\left(\frac{\partial L_{rel}}{\partial \dot{\theta}}\right) = 0 \to \mu r^2\dot{\theta} = l \text{ fasti}$$

Hreyfijöfnur

$$\frac{\partial L_{rel}}{\partial r} - \frac{d}{dt}\left(\frac{\partial L_{rel}}{\partial \dot{r}}\right) = 0, \quad \mu r^2\dot{\theta} = l$$

$$\mu r\dot{\theta}^2 - kr - \mu\ddot{r} = 0 \to \boxed{\begin{gathered}\ddot{r} - \frac{l^2}{\mu^2 r^3} + \frac{k}{\mu}r = 0 \\ l = \mu r^2\dot{\theta} \text{ fasti} \\ \bar{P}_{cm} = M\dot{\bar{R}} \text{ fasti}\end{gathered}}$$

Fall Hamiltons

$$L = \frac{M}{2}|\dot{\bar{R}}|^2 + \frac{\mu}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2\right\} - \frac{k}{2}r^2$$

$$\boxed{\begin{gathered}\bar{P}_{cm} = \frac{\partial L}{\partial \dot{\bar{R}}} = M\dot{\bar{R}}, \quad p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r} \\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} \quad \text{alskriðþungar}\end{gathered}}$$

$$H = \bar{P}_{cm}\cdot\dot{\bar{R}} + p_\theta\dot{\theta} + p_r\dot{r} - L$$

$$= M|\dot{\bar{R}}|^2 + l\dot{\theta} + \mu\dot{r}^2 - \frac{M}{2}|\dot{\bar{R}}|^2 - \frac{\mu\dot{r}^2}{2} - \frac{\mu(r\dot{\theta})^2}{2} + \frac{k}{2}r^2$$

$$\boxed{= \frac{|\bar{P}_{cm}|^2}{2M} + \frac{p_r^2}{2\mu} + \frac{l^2}{2\mu r^2} + \frac{k}{2}r^2}$$

$$\boxed{\begin{gathered}p_\theta = l \text{ fasti} \\ \bar{P}_{cm} = m\dot{\bar{R}} \text{ fasti} \\ \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu} \\ -\dot{p}_r = \frac{\partial H}{\partial r} = -\frac{l^2}{\mu r^3} + kr\end{gathered}}$$

Hreyfijöfnur Hamiltons

11

Dæmi 4

Ögninni er lýst með

$$L = \frac{m}{2}\left\{\dot r^2 + (r\dot\theta)^2 + \dot z^2\right\} - mgz$$

og skorðunum $r^2 = 4az \to 2r\dot r = 4a\dot z \to \dot z = \frac{r\dot r}{2a}$

Notum beint í L til að fækka alhnitum

$$\to L = \frac{m}{2}\left\{\dot r^2 + (r\dot\theta)^2 + \left(\frac{r\dot r}{2a}\right)^2\right\} - mg\frac{r^2}{4a}$$

$$= \frac{m}{2}\left\{\left[1+\frac{1}{4}\left(\frac{r}{a}\right)^2\right]\dot r^2 + (r\dot\theta)^2\right\} - mg\frac{r^2}{4a}$$

Miðlægt mætti, óháð θ, sem er þá rásuð breyta

$\to mr^2\dot\theta = l$ fasti

því er aðeins eftir eitt alhnit r

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Euler-Lagrange

$$\frac{\partial}{\partial L} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = 0$$

$$\to \frac{m}{4a^2} r\dot r^2 + mr\dot\theta^2 - \frac{mg}{2a}r - \frac{d}{dt}\left\{m\left[1+\frac{1}{4}\left(\frac{r}{a}\right)^2\right]\dot r\right\} = 0$$

$$\to \boxed{m\left[1+\frac{1}{4}\left(\frac{r}{a}\right)^2\right]\ddot r + \frac{mr\dot r^2}{4a^2} + \frac{mg}{2a}r - \frac{l^2}{mr^3} = 0}$$

sem er hreyfijafna agnarinnar. Finnum hringlaga braut $\to \ddot r = 0, \dot r = 0, r = r_0$

$$\to \frac{mg}{2a}r_0 - \frac{l^2}{mr_0^3} = 0 \to l^2 = \frac{m^2 g}{2a} r_0^4$$

en $l^2 = m^2 r_0^4 \dot\theta^2 \to \boxed{m^2 r_0^4 \dot\theta^2 = \frac{m^2 g}{2a} r_0^4}$

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Smáar sveiflur, truflun um hringbraut

$$r = r_0 + \delta, \quad \dot r = \dot\delta, \quad \ddot r = \ddot\delta$$

$$(r_0+\delta)^2 \simeq r_0^2 + 2r_0\delta + \dots$$

$$\frac{1}{(r_0+\delta)^3} \simeq \frac{1}{r_0^3}\left\{1 - 3\frac{\delta}{r_0} + \dots\right\}$$

$$r\dot r^2 = (r_0+\delta)\dot\delta^2 \to 0$$

$$r^2\ddot r \simeq (r_0+\delta)^2\ddot\delta = (r_0^2 + 2r_0\delta + \dots)\ddot\delta \simeq r_0^2\ddot\delta$$

því verður hreyfijafnan

$$m\left\{1+\frac{r_0^2}{4a^2}\right\}\ddot\delta + \frac{mg}{2a}(r_0+\delta) - \frac{l^2}{mr_0^3}\left\{1 - 3\frac{\delta}{r_0}\right\} = 0$$

Notum jafnvægisskilyrðið

$$\frac{mg}{2a}r_0 - \frac{l^2}{mr_0^3} = 0$$

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til að umskrifa hreyfijöfnuna sem

$$m\left\{1+\frac{r_0^2}{4a^2}\right\}\ddot\delta + \left\{\frac{mg}{2a} + \frac{3l^2}{mr_0^4}\right\}\delta = 0$$

Jafnvægisskilyrðið gefur líka,

$$l^2 = \frac{m^2 g}{2a} r_0^4$$

og skilgreinum

$$z_0 \equiv \frac{r_0^2}{4a^2}$$

Þá fæst hreyfijafnan

$$\boxed{\ddot\delta + \left[\frac{2g}{a+z_0}\right]\delta = 0}$$

Þessi línulega nálgun hreyfijöfnunnar gefur grunntíðnina fyrir smáar sveiflur

$$\boxed{\omega = \sqrt{\frac{2g}{a+z_0}}}$$