(1)

$$m\ddot{x} = -\lambda\,\mathrm{sgn}(\dot x)\,|\dot x|^\alpha$$

$t>0,\ \dot x>0$ gerum ráð fyrir

$\dot x(t=0)=v_0 \qquad x(t=0)=x_0$

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{m}\left(\frac{dx}{dt}\right)^\alpha$$

Breytum í jöfnu fyrir v

$$\frac{dv}{dt} = -\frac{\lambda}{m}v^\alpha$$

$$\frac{dv}{v^\alpha} = -\frac{\lambda}{m}dt$$

heildum

$$\int_{v_0}^{v}\frac{dv'}{(v')^\alpha} = -\frac{\lambda}{m}\int_0^t dt$$

(1)

$$-\frac{v^{1-\alpha}}{\alpha-1}+\frac{v_0^{1-\alpha}}{\alpha-1} = -\frac{\lambda}{m}t$$

$$v^{1-\alpha}-v_0^{1-\alpha} = \frac{\lambda}{m}(\alpha-1)t$$

$$v^{1-\alpha} = v_0^{1-\alpha}+\frac{\lambda}{m}(\alpha-1)t = (\alpha-1)\left(-\frac{v_0^{1-\alpha}}{1-\alpha}+\frac{\lambda t}{m}\right)$$

$$\boxed{v = \left\{(\alpha-1)\left(-\frac{v_0^{1-\alpha}}{1-\alpha}+\frac{\lambda t}{m}\right)\right\}^{\frac{1}{1-\alpha}}}$$

$$\frac{dx}{dt} = \left\{(\alpha-1)\left(-\frac{v_0^{1-\alpha}}{1-\alpha}+\frac{\lambda t}{m}\right)\right\}^{\frac{1}{1-\alpha}}$$

(2)

$$\int_{x_0}^{x}dx' = \int_0^t dt'\left\{(\alpha-1)\left(-\frac{v_0^{1-\alpha}}{1-\alpha}+\frac{\lambda t'}{m}\right)\right\}^{\frac{1}{1-\alpha}}$$ heildum

$$x-x_0 = \frac{m}{\lambda}\frac{(\alpha-1)^{\frac{1}{1-\alpha}}}{\left(\frac{2-\alpha}{1-\alpha}\right)}\left(-\frac{v_0^{1-\alpha}}{1-\alpha}+\frac{\lambda t'}{m}\right)^{\overbrace{\frac{1+1-\alpha}{1-\alpha}}^{\frac{2-\alpha}{1-\alpha}}}\Bigg|_0^t$$

$$= -\frac{m}{\lambda(2-\alpha)}\left\{(\alpha-1)\left(-\frac{v_0^{1-\alpha}}{1-\alpha}+\frac{\lambda t'}{m}\right)\right\}^{\frac{2-\alpha}{1-\alpha}}\Bigg|_0^t$$

$$= +\frac{m}{\lambda}\frac{v_0^{2-\alpha}}{(2-\alpha)} - \frac{m}{\lambda(2-\alpha)}\left\{(\alpha-1)\left(\frac{v_0^{1-\alpha}}{(\alpha-1)}+\frac{\lambda t}{m}\right)\right\}^{\frac{2-\alpha}{1-\alpha}}$$

$$= \frac{m}{\lambda(2-\alpha)}\left[v_0^{2-\alpha} - \left\{v_0^{1-\alpha}+\frac{\lambda t}{m}(\alpha-1)\right\}^{\frac{2-\alpha}{1-\alpha}}\right]$$

(3)

því fæst

$$x-x_0 = \frac{m v_0^{2-\alpha}}{\lambda(2-\alpha)}\left[1-\left\{1+\frac{\lambda t(\alpha-1)}{m v_0^{1-\alpha}}\right\}^{\frac{2-\alpha}{1-\alpha}}\right]$$

nálgum þegar t er lítið, þ.e. ef $\frac{\lambda t(\alpha-1)}{m v_0^{1-\alpha}} \ll 1$

(við þurfum alltaf viðmið, sem fæst með því að gera breyturnar víddarlausar)

notum Taylor $(1+x)^\beta \simeq 1+\beta x+\frac{(\beta^2-\beta)}{2}x^2+\dots\dots$

$$x-x_0 \simeq \frac{m v_0^{2-\alpha}}{\lambda(2-\alpha)}\left\{-\frac{(2-\alpha)}{(1-\alpha)}\frac{\lambda t(\alpha-1)}{m v_0^{1-\alpha}} - \frac{\left(\left(\frac{2-\alpha}{1-\alpha}\right)^2-\frac{(2-\alpha)}{(1-\alpha)}\right)}{2}\frac{\lambda^2 t^2(\alpha-1)^2}{m^2 v_0^{2(1-\alpha)}}+\dots\right\}$$

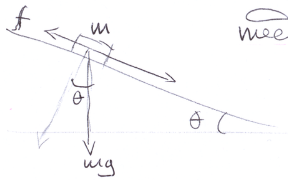
$$\simeq v_0 t - \frac{\lambda}{2m}v_0^\alpha t^2+\dots$$

fyrir lítið t er eins og kraftur λv_0^α sé á móti hreyfingunni

sjá M. Razavy, Dissipative classical and Quantum systems 2nd Ed. 1. kafla

(2-15) (4)

Skoðum hreyfingu agnar í föstu þyngdarsviði og með viðnámskraft $f = kmv^2$



Hve langan tíma þarf ögnin til að komast d úr kyrrstöðu?

Hreyfijafnan er

$$m\ddot x = mg\sin\theta - kmv^2 \;\to\; \boxed{\frac{dv}{dt} = g\sin\theta - kv^2} \quad (*)$$

eða

$$dv = g\sin\theta\cdot dt - kv^2\,dt = \left\{g\sin\theta - kv^2\right\}dt$$

og

$$\frac{dv}{\left\{g\sin\theta-kv^2\right\}} = dt$$

höfum aðgreint breyturnar t og v og getum því heildað beint upphafsstilyrði

$$v(t=0) = v(0) = 0$$

$$\int_0^v \frac{dv'}{g\sin\theta - k(v')^2} = \int_0^t dt'$$

Hröðun agnarinnar eykst

$$\to \frac{dv}{dt} > 0$$

því sést frá (*) að

$$g\sin\theta > kv^2$$

$$\frac{1}{\sqrt{kg\sin\theta}}\cdot \mathrm{Artanh}\left[\frac{\sqrt{k}\,v'}{\sqrt{g\sin\theta}}\right]\Bigg|_0^v = t$$

$$\to t = \frac{\mathrm{Artanh}\left\{\frac{\sqrt{k}\,v}{\sqrt{g\sin\theta}}\right\}}{\sqrt{kg\sin\theta}}$$

$$\mathrm{Artanh}\left[\frac{\sqrt{k}\,v}{\sqrt{g\sin\theta}}\right] = t\sqrt{kg\sin\theta}$$

$$\to \frac{\sqrt{k}\,v}{\sqrt{g\sin\theta}} = \tanh\left(t\sqrt{kg\sin\theta}\right)$$

og

$$v = \sqrt{\frac{g\sin\theta}{k}}\tanh\left(t\sqrt{kg\sin\theta}\right)$$

eða

$$\frac{dx}{dt} = \sqrt{\frac{g\sin\theta}{k}}\tanh\left(t\sqrt{kg\sin\theta}\right)$$

(5)

þetta getum við heildað

$$\int_0^d dx = \int_0^t dt' \sqrt{\frac{g\sin\theta}{k}}\tanh\left(t'\sqrt{kg\sin\theta}\right)$$

$$d = \sqrt{\frac{g\sin\theta}{k}}\,\frac{\ln\left\{\cosh\left(t\sqrt{kg\sin\theta}\right)\right\}}{\sqrt{kg\sin\theta}}$$

$$\to d = \frac{1}{k}\ln\left\{\cosh\left(t\sqrt{kg\sin\theta}\right)\right\}$$

Henni er hægt að snúa við

$$\exp\{kd\} = \cosh\left(t\sqrt{kg\sin\theta}\right) \to$$

$$t\sqrt{kg\sin\theta} = \mathrm{ArCosh}\{\exp(kd)\}$$

eða

$$\boxed{t = \frac{\mathrm{ArCosh}\{\exp(kd)\}}{\sqrt{kg\sin\theta}}}$$

(6)

$$\lim_{kd\to 0} t = \frac{\sqrt{2\cdot kd}}{\sqrt{kg\sin\theta}} = \sqrt{\frac{2d}{g\sin\theta}}$$

d fasti
$k \to 0$

þekkt lausn þegar enginn núningur er

(7)

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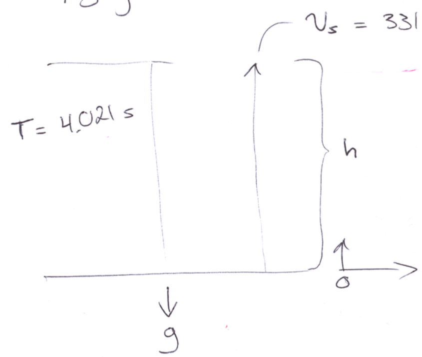
Þyngdarhröðun



Lóðrétt fall $m\ddot{y} = -mg$

lausn

$$y = y_0 + \dot{y}_0 t - \frac{1}{2}gt^2$$

$$\left.\begin{array}{l} y(0) = h \\ \dot{y}(0) = 0 \end{array}\right\} \quad 0 = h - \frac{1}{2}gt^2$$

$$\to t_f = \sqrt{\frac{2h}{g}}$$

Hljóð upp $t_s = \frac{h}{v_s}$ $\to$ heildartími þ.t. hljóð heyrist er

$$T = t_s + t_f = \frac{h}{v_s} + \sqrt{\frac{2h}{g}}$$

þarf nú að leysa

gefið

$$\boxed{\frac{h}{v_s} + \sqrt{\frac{2h}{g}} - T = 0}$$

Annars stigs jafna fyrir breytuna $\sqrt{h}$

(8)

$$\sqrt{h} = \frac{-\sqrt{\frac{2}{g}} \pm \sqrt{\frac{2}{g} + \frac{4T}{v_s}}}{2/v_s} = \frac{v_s}{\sqrt{2g}}\left[-1 \pm \sqrt{1 + \frac{2gT}{v_s}}\right]$$

h er jákvæð stærð $\rightarrow$ tökum jákvæðu rótina

$\rightarrow h = 71.063$ m ef $g = 9.81\ m/s^2$

(2-40)

Ögn á braut

$$x(t) = A(2\alpha t - \sin(\alpha t))$$
$$y(t) = A(1 - \cos(\alpha t))$$

Hraðinn er alltaf snertill ~~við~~ brautina

Skilgreinum $\bar{v}(t) = v(t)\bar{T}(t)$

snertill brautar

$$\rightarrow \bar{a}(t) = \frac{d\bar{v}}{dt} = \frac{dv}{dt}\cdot\bar{T} + v\frac{d\bar{T}}{dt}$$

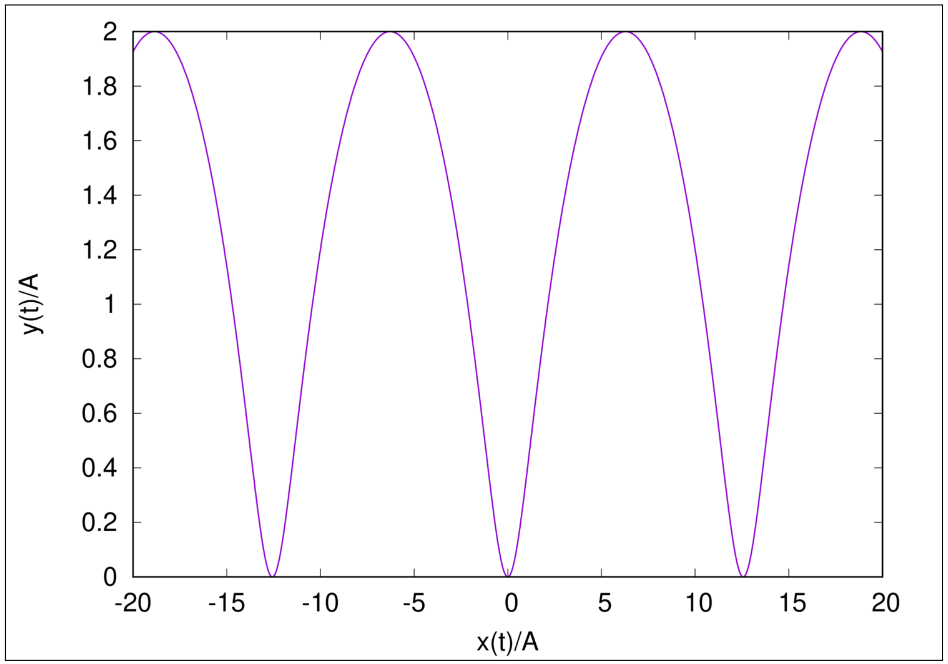
sjá mynd á næstu síðu

$$= a_t\bar{T} + a_n\bar{N}$$

$$\bar{T}\cdot\bar{T} = 1 \rightarrow \frac{d}{dt}(\bar{T}\cdot\bar{T}) = 0 = \bar{T}\cdot\frac{d\bar{T}}{dt} + \frac{d\bar{T}}{dt}\cdot\bar{T}$$
$$= 2\bar{T}\cdot\bar{N}$$

$\bar{N}$ og $\bar{T}$ eru hornréttir

Einingarvigrar $\bar{T}(t) = \frac{\dot{\bar{r}}(t)}{|\dot{\bar{r}}(t)|}$, $\bar{N} = \frac{\dot{\bar{T}}(t)}{|\dot{\bar{T}}(t)|}$



$$\dot{x}(t) = A(2\alpha - \alpha\cos(\alpha t)) = A\alpha(2 - \cos(\alpha t))$$
$$\dot{y}(t) = A\alpha\sin(\alpha t)$$

$$\rightarrow v = \sqrt{\dot{x}^2 + \dot{y}^2} = A\alpha\sqrt{(2-\cos(\alpha t))^2 + \sin^2(\alpha t)}$$
$$= A\alpha\sqrt{5 - 4\cos(\alpha t)}$$

$$\rightarrow a_t = \frac{dv}{dt} = \frac{2A\alpha^2\sin(\alpha t)}{\sqrt{5 - 4\cos(\alpha t)}}$$

Varúð $\dot{\bar{T}} \neq \bar{N} \rightarrow a_n \neq v$ ($\dot{\bar{T}}$ er ekki stöðlaður)

en $a^2 = a_t^2 + a_n^2$, og

$$a = \sqrt{(\ddot{x}(t))^2 + (\ddot{y}(t))^2} = A\alpha^2$$

$$\rightarrow a_n = \sqrt{a^2 - a_t^2}$$

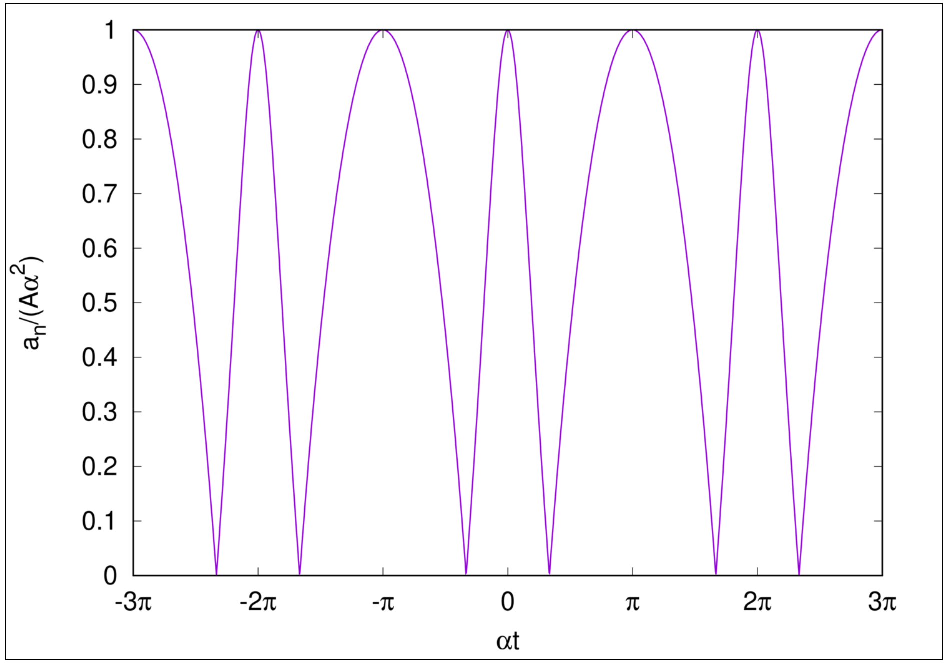
(13)

$$a_n = \sqrt{A^2\alpha^4 - \frac{4A^2\alpha^4\sin^2(\alpha t)}{5-4\cos(\alpha t)}} = A\alpha^2\sqrt{1-\frac{4\sin^2(\alpha t)}{5-4\cos(\alpha t)}}$$

$$= A\alpha^2\sqrt{\frac{5-4\cos(\alpha t)-4\sin^2(\alpha t)}{5-4\cos(\alpha t)}} = A\alpha^2\sqrt{\frac{1-4\cos(\alpha t)+4\cos^2(\alpha t)}{5-4\cos(\alpha t)}}$$

$$= A\alpha^2\sqrt{\frac{(1-2\cos(\alpha t))^2}{5-4\cos(\alpha t)}} = A\alpha^2\frac{|1-2\cos(\alpha t)|}{\sqrt{5-4\cos(\alpha t)}}$$

sjá graf á næstu síðu, sem sýnir max-gildi a_n í punktunum $\alpha t = n\pi$, $n \in \mathbb{Z}$



2-52

(15)

$$U(x) = U_0\left\{2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4\right\}$$ (sjá mynd á næstu síðu)

$U_0, a > 0$

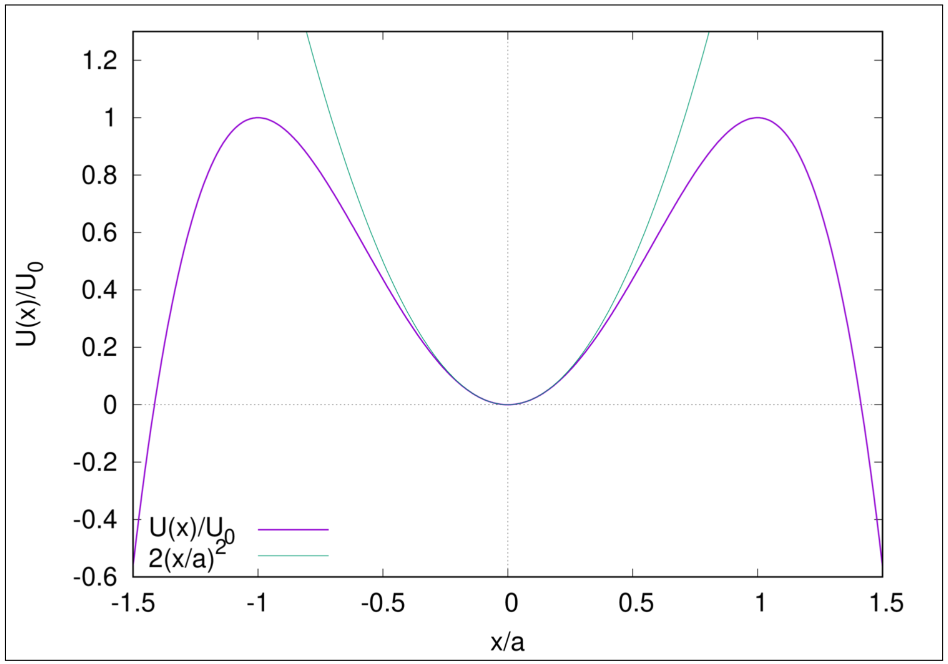
a) Krafturinn vegna mættisins $U(x)$

$$F(x) = -\frac{d}{dx}U(x) = -\frac{U_0}{a}\left\{4\left(\frac{x}{a}\right) - 4\left(\frac{x}{a}\right)^3\right\}$$

$$= -4\left(\frac{U_0}{a}\right)\left(\frac{x}{a}\right)\left\{1-\left(\frac{x}{a}\right)^2\right\}$$

b) stöðugt staðbundið lágmark þ. $x = 0$ því þar er $\frac{\partial^2 U}{\partial x^2} > 0$

óstöðug hámörk þegar $\left(\frac{x}{a}\right) = \pm$ því þar gildir að $\frac{\partial^2 U}{\partial x^2} < 0$



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c) hver er horntíðni sveiflna ognar um $x = 0$

Þar er mættið
$$U(x) \approx U_0 \frac{2}{a^2} x^2$$

Ein vidur heimtóna sveifill er við hreyfijöfnu,
$$\ddot{x} + \omega^2 x = 0, \qquad \omega = \sqrt{\frac{k}{m}}$$

mættið $U(x) = \frac{1}{2}kx^2$ og kraftinn $F(x) = -kx$

Því er horntíðnin hér
$$\omega = \sqrt{\frac{4U_0}{ma^2}}$$

d) minnsti hraði fyrir ögn til að sleppa frá lágmarki í $x=0$. Þá verður hreyfiorkan í $x=0$ jöfn mættisorku í $x = \pm a$
$$\frac{m v_m^2}{2} = U(a) = U_0 \longrightarrow v_{min} = \sqrt{\frac{2U_0}{m}}$$

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e) Ef í $t=0$ $x(0)=0$ og $v = v_{min}$

Hér er textinn ekki skýr, en ég geri ráð fyrir að höfundur vilji vita hvernig þessi markhreyfing lítur út frá $x=0$ að $x=a$

Orkan er varðveitt, í $x=0$ er
$$E_{min} = \frac{m v_{min}^2}{2}, \qquad E_{min} = U_0$$

og í hverjum punkti
$$E_{min} = \frac{mv^2}{2} + U(x) = \frac{mv^2}{2} + U_0\left\{2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4\right\}$$
$$\longrightarrow U_0 = \frac{mv^2}{2} + U_0\left\{2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4\right\}$$
$$\longrightarrow \frac{mv^2}{2} = U_0 - U_0\left\{2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4\right\} = U_0\left\{1 - 2\left(\frac{x}{a}\right)^2 + \left(\frac{x}{a}\right)^4\right\}$$

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$$v = \sqrt{\frac{2U_0}{m}\left\{1 - 2\left(\frac{x}{a}\right)^2 + \left(\frac{x}{a}\right)^4\right\}} = v(x)$$

en $v = \frac{dx}{dt} \longrightarrow dt = \frac{dx}{v(x)}$
$$\longrightarrow t = \sqrt{\frac{m}{2U_0}} \int_0^x \frac{dx'}{\sqrt{1 - 2\left(\frac{x'}{a}\right)^2 + \left(\frac{x'}{a}\right)^4}} = \sqrt{\frac{ma^2}{2U_0}} \int_0^{\frac{x}{a}} \frac{du}{\sqrt{1-2u^2+u^4}}$$

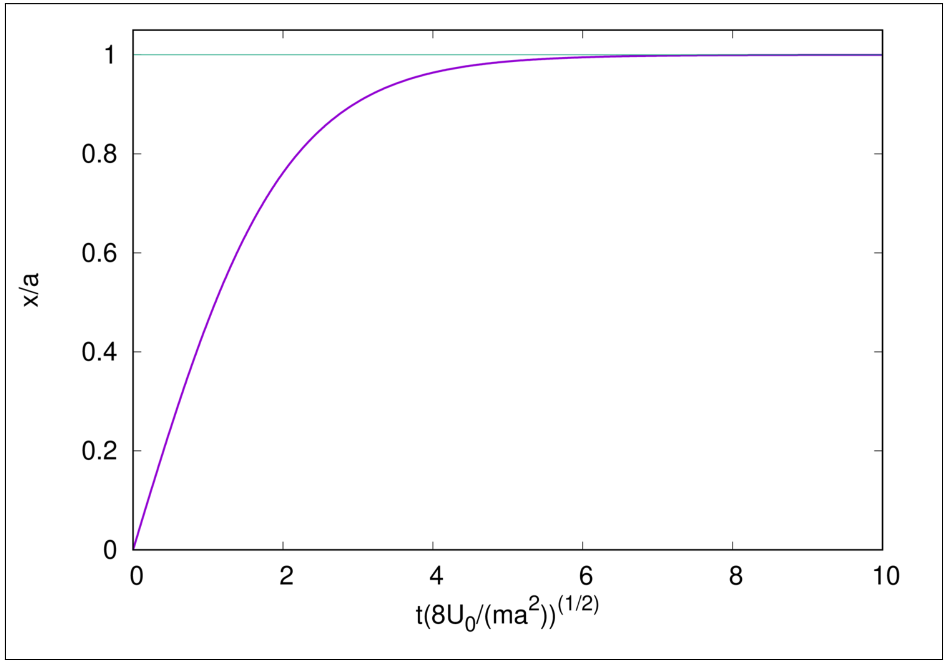
þegar $\frac{x}{a} < 1$
$$t = \sqrt{\frac{ma^2}{8U_0}} \left\{\ln \frac{1+\frac{x}{a}}{1-\frac{x}{a}}\right\} \longrightarrow \frac{t}{\sqrt{\frac{ma^2}{8U_0}}} = \ln\left(\frac{a+x}{a-x}\right)$$

20

og því
$$\exp\left\{t\sqrt{\frac{8U_0}{ma^2}}\right\} = \frac{a+x}{a-x}$$
$$\longrightarrow x(t) = a\frac{\left\{\exp\left(t\sqrt{\frac{8U_0}{ma^2}}\right) - 1\right\}}{\left\{\exp\left(t\sqrt{\frac{8U_0}{ma^2}}\right) + 1\right\}}$$

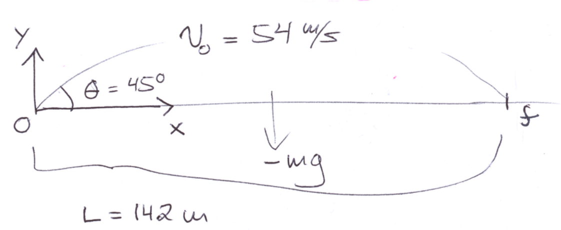
með $\lim_{t\to\infty} x(t) = a$ ← ögnin tekur óendanlegan tíma að komast í $x=a$

sjá mynd á næstu síðu



Graskev $m = 5\,kg$, $F = -kmv$

ákvarða k



Hreyfijöfnur

x-þáttur : $ma_x = F_x = -mkv_x$

eða $m\frac{dv_x}{dt} = -mkv_x \to \boxed{\frac{dv_x}{v_x} = -k\,dt}$ (1)

einnig er hægt að fá

$\frac{dv_x}{dt} = -k\frac{dx}{dt} \to \boxed{dv_x = -k\,dx}$ (2)

Heildum (2)

$$\int_{v_{x0}}^{v_{xf}} dv_x' = -k\int_0^{x_f} dx'$$

$$v_{xf} - v_{x0} = -k x_f$$

Heildum (1)

$$\int_{v_{x0}}^{v_{xf}} \frac{dv_x'}{v_x'} = -k\int_0^{t_f} dt$$

$$\ln\left\{\frac{v_{xf}}{v_{x0}}\right\} = -kt_f$$

$$v_{xf} = v_{x0}\exp(-kt_f)$$

$$\boxed{x_f = \frac{v_{x0}}{k}\left[1-\exp(-kt_f)\right]}\quad (*)$$

y-þáttur :

$$m\frac{dv_y}{dt} = -mg - mkv_y$$

$$\to \frac{dv_y}{dt} = -g - kv_y$$

$$\to \boxed{\frac{dv_y}{g+kv_y} = -dt}\quad (3)$$

Einnig má umrita hreyfijöfnuna sem

$$\frac{dv_y}{dt} = -g - k\frac{dy}{dt} \to \frac{dv_y}{dt} + \frac{k\,dy}{dt} = -g$$

eða $dv_y + k\,dy = -g\,dt$ notum $\frac{dy}{dt} = v_y$ $\to$ $dt = \frac{dy}{v_y}$

$$dv_y + k\,dy = -g\frac{dy}{v_y} \to dv_y = -k\,dy - g\frac{dy}{v_y}$$

$$= -\left(k + \frac{g}{v_y}\right)dy$$

$$\to \frac{dy}{v_y} = -\frac{dv_y}{(kv_y+g)}$$

eða $\boxed{dy = -\frac{v_y\,dv_y}{(kv_y+g)}}$ (4)

Heildum ③

$$\int_{v_{y0}}^{v_{yf}} \frac{dv_y}{g+kv_y} = -\int_0^{t_f} dt$$

sem gefur

$$\frac{1}{k}\ln\left(\frac{g+kv_{yf}}{g+kv_{y0}}\right) = -t_f$$

eða

$$\boxed{g+kv_{yf} = (g+kv_{y0})\exp(-kt_f)} \quad ⑤$$

heildum ④

$$\int_0^0 dy = -\int_{v_{y0}}^{v_{yf}} \frac{v_y\, dv_y}{(kv_y+g)}$$

$$\boxed{0 = \frac{g}{k^2}\ln\left\{\frac{kv_{yf}+g}{kv_{y0}+g}\right\} - \left(\frac{v_{yf}-v_{y0}}{k}\right)} \quad ⑥$$

Notum ⑤ og ⑥ til að losna við v_{yf}, sem er óþekkt stærð.

$$\frac{g}{k^2}\ln\left(\frac{kv_{yf}+g}{kv_{y0}+g}\right) = -\frac{g}{k}t_f$$

$$0 = -\frac{g}{k}t_f - \frac{v_{yf}-v_{y0}}{k}$$

$$\frac{g}{k}t_f = -\frac{v_{yf}}{k} + \frac{v_{y0}}{k}$$

$$\frac{g}{k}t_f = +\frac{v_{y0}}{k} - \frac{1}{k^2}(g+kv_{y0})e^{-kt_f} + \frac{g}{k^2}$$

$$= +\frac{1}{k^2}(g+kv_{y0})\left(1-e^{-kt_f}\right)$$

$$\rightarrow \boxed{\left(1-e^{-kt_f}\right) = \frac{gkt_f}{g+kv_{y0}}} \quad ⑦$$

tvær jöfnur, tvær óþekktar stærðir t_f og k

leysum saman með (*):

$$\boxed{x_f = \frac{v_{x0}}{k}\left[1-e^{-kt_f}\right]} \quad ⑧$$

(x_f, v_{0x}, v_{0y} þekkt)

óháðar

sem líka hefur fengist

$$e^{-kt_f} = 1 - \frac{x_f k}{v_{x0}}$$

$$-kt_f = \ln\left(1-\frac{x_f k}{v_{x0}}\right) \quad ⑨$$

⑦ og ⑧

$$\frac{x_f k}{v_{x0}} = \frac{gkt_f}{g+kv_{y0}}$$

⑨

$$\frac{x_f k}{v_{x0}} = -\frac{g}{g+kv_{y0}}\ln\left(1-\frac{x_f k}{v_{x0}}\right)$$

ein jafna, ein óþekkt stærð, k

$$-\frac{x_f k}{v_{x0}}\frac{(g+kv_{y0})}{g} = \ln\left(1-\frac{x_f k}{v_{x0}}\right)$$

þ. $k \to 0$ stefnir hægri hliðin einnig á x_f, eins og ætti að vera

$$\rightarrow \exp\left\{-\frac{x_f k(g+kv_{y0})}{v_{x0}\cdot g}\right\} = 1 - \frac{x_f k}{v_{x0}}$$

$$\boxed{x_f = \frac{v_{x0}}{k}\left\{1-\exp\left[-\frac{x_f k(g+kv_{y0})}{v_{x0}\cdot g}\right]\right\}}$$

ólínuleg jafna sem við leysum tölulega

$$\underbrace{\qquad}_{x_f k = F(k)}$$

$v_0 = 54$ m/s, $x_f = 142$ m, $v_{0y} = 54\cdot\cos\left(\frac{\pi}{4}\right) \approx 38.2$ m/s

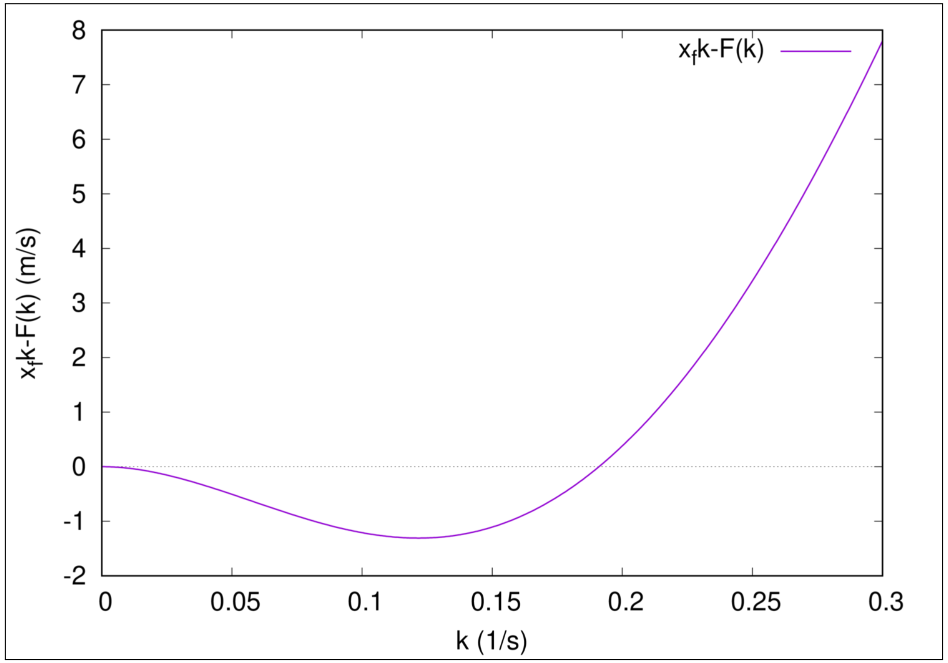
$\theta = 45°$, $g = 9.81$ m/s^2

$x_f k = F(k)$

leysum

$$x_f k - F(k) = 0$$

Grafið á næstu síðu sýnir að $k=0$ er lausn, eins og við höfðum séð áður, en líka sést önnur lausn sem wxMaxima gefur sem $k = 0{,}19105\ldots\ s^{-1}$



(%i2) find_root(142·x−38.2·(1−exp(−(142·x·(9.81+x·38.2))/(38.2·9.81)))=0, x, 0.1, 0.3);

(%o2) 0.1910528801334695

Created with wxMaxima.

3-15

1

$$x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$$

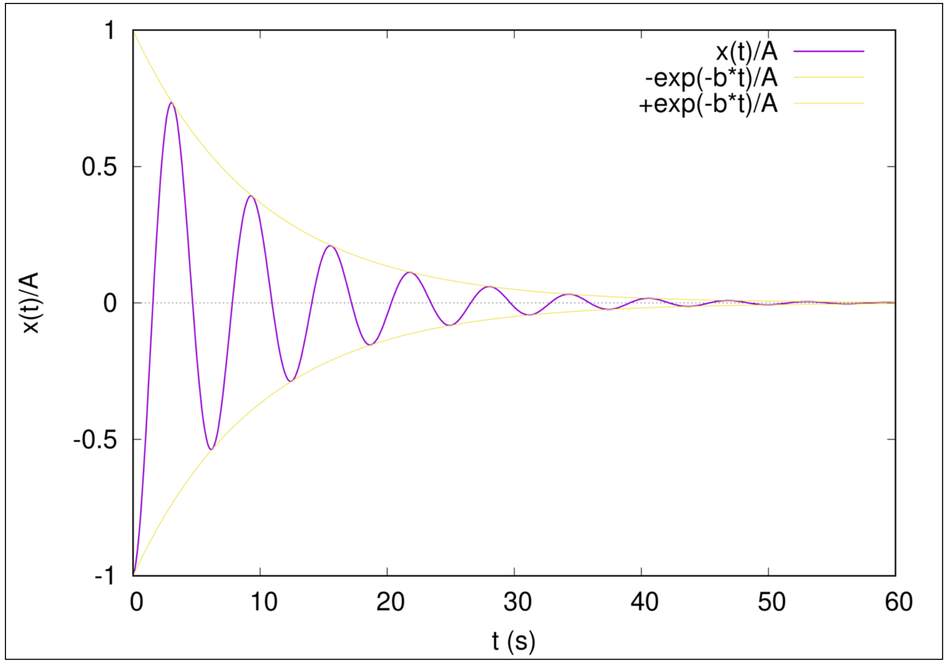
$$\dot{x}(t) = -A e^{-\beta t} \left\{ \beta \cos(\omega_1 t - \delta) + \omega_1 \sin(\omega_1 t - \delta) \right\}$$

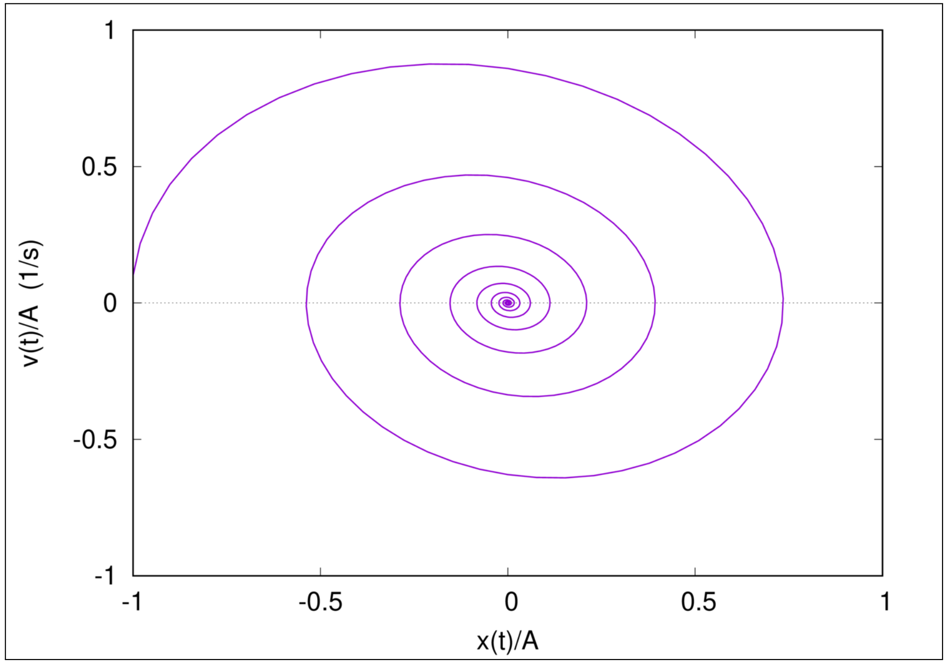
$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ $\quad$ $A = 1, \ \omega_0 = 1$ rad/s

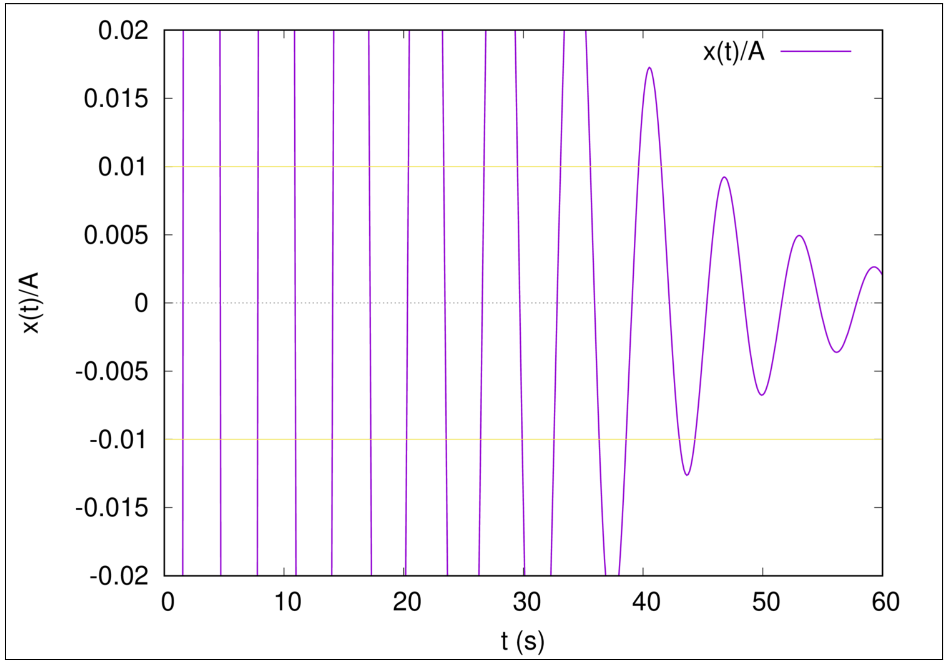
$\beta = 0,1 \ s^{-1}$ $\quad$ $\delta = \pi$ rad

Hve margar sveiflur áður en $\frac{x}{A} < 10^{-2}$

sjá næstu 3 gröf þar sem $t \in [0, 60]$ s







5

3-42 Ódeyfður hreintóna sveifill.

$$m\ddot{x} + m\omega_0^2 x = \Theta(t) F_0 \sin(\omega t)$$

upphafsskilyrði

$x(0) = 0$

$v(0) = 0$

Finna lausn þegar $\omega \to \omega_0$

Almenn lausn óhliðruðu jöfnunnar er

$$x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

ein sérstök lausn $C\sin(\omega t) \to (-m\omega^2 + m\omega_0^2) C = F_0$ (*)

$$\to x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) + C\sin(\omega t)$$

$$0 = B\cos(\omega_0 t) \to B = 0$$

$$0 = A\omega_0 + C\omega \to$$

6

(*) $\to C = \dfrac{F_0}{m(\omega_0^2 - \omega^2)}$ og því $A\omega_0 = -\dfrac{F_0\omega}{m(\omega_0^2-\omega^2)}$

$$A = -\frac{F_0\omega}{m\omega_0}\frac{1}{(\omega_0^2-\omega^2)}$$

$$\to x(t) = -\frac{F_0\omega}{m\omega_0}\frac{\sin(\omega_0 t)}{(\omega_0^2-\omega^2)} + \frac{F_0\sin(\omega t)}{m(\omega_0^2-\omega^2)}$$

$$= \frac{F_0}{m\omega_0}\frac{1}{(\omega_0^2-\omega^2)}\left\{\omega_0\sin(\omega t) - \omega\sin(\omega_0 t)\right\}$$

b)

$$x(t) = \frac{F_0}{m\omega_0(\omega_0+\omega)}\left\{\frac{\omega_0\sin(\omega t) - \omega\sin(\omega_0 t)}{\omega_0 - \omega}\right\}$$

7

$$\lim_{\omega\to\omega_0}\left\{\frac{\omega_0\sin(\omega t)-\omega\sin(\omega_0 t)}{\omega_0-\omega}\right\} = \sin(\omega_0 t)-(\omega_0 t)\cos(\omega_0 t)$$

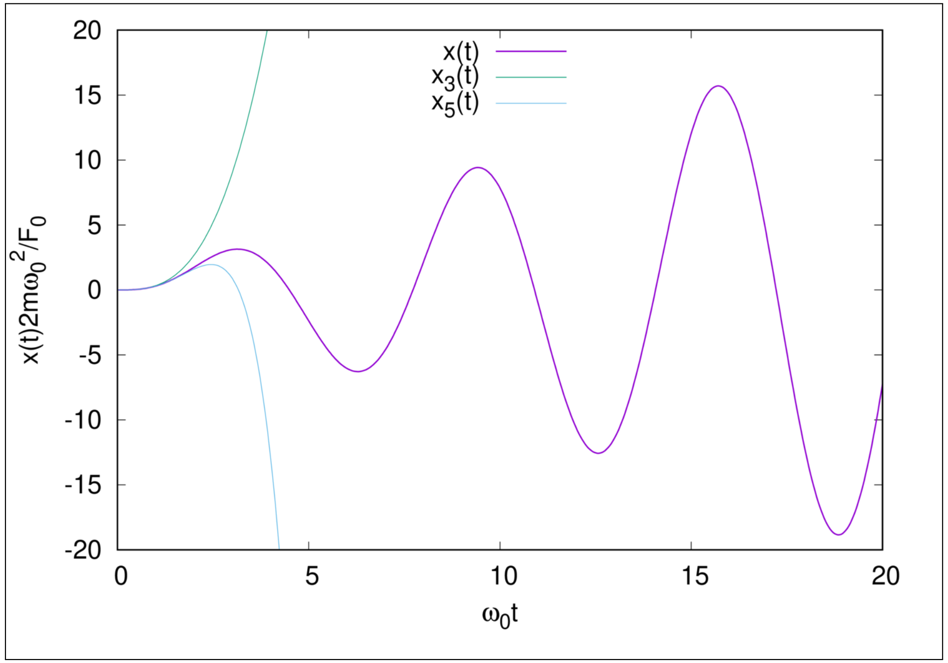
$\to$ fyrir $\omega \to \omega_0$

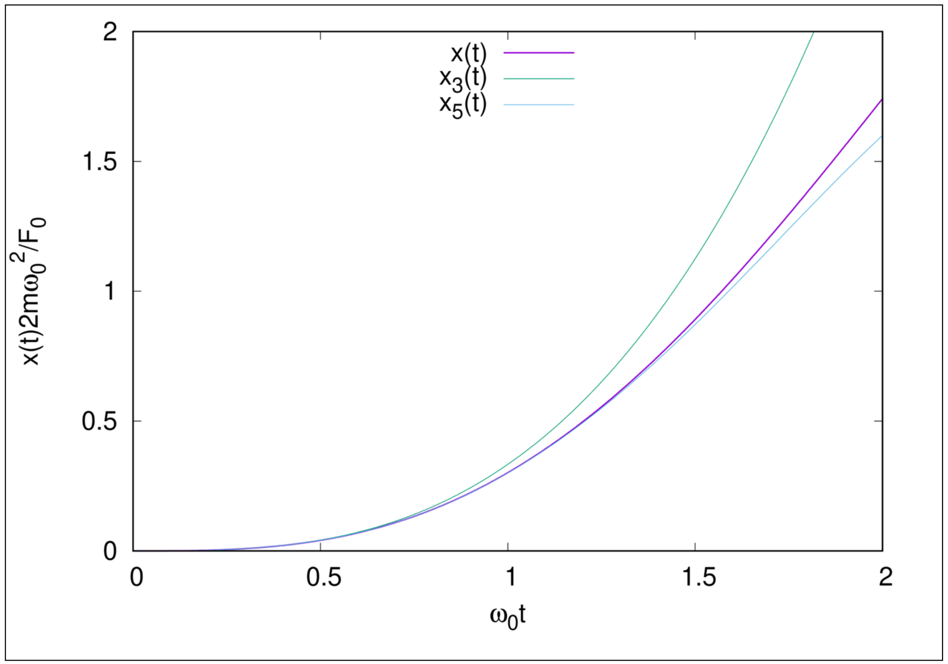
$$x(t) = \frac{F_0}{2m\omega_0^2}\left\{\sin(\omega_0 t)-(\omega_0 t)\cos(\omega_0 t)\right\}$$

Síðan má sjá

$$x(t) = \underbrace{\underbrace{\frac{F_0}{2m\omega_0^2}\left\{\frac{(\omega_0 t)^3}{3}\right.}_{x_3(t)} - \frac{(\omega_0 t)^5}{30}}_{x_5(t)} \left. + \cdots \right\}$$

þegar $(\omega_0 t) \to 0$





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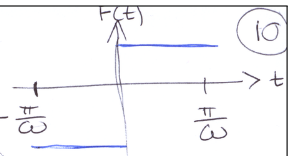
3-28

Fourier röð fyrir

$$F(t) = \begin{cases} -1 & -\frac{\pi}{\omega} < t < 0 \\ +1 & 0 < t < \frac{\pi}{\omega} \end{cases}$$



Oddstætt fall

$$F(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} dt'\, F(t')\sin(n\omega t) \qquad n$$

$$= \frac{\omega}{\pi}\left[-\int_{-\pi/\omega}^{0} dt' \sin(n\omega t) + \int_{0}^{\pi/\omega} dt' \sin(n\omega t)\right]$$

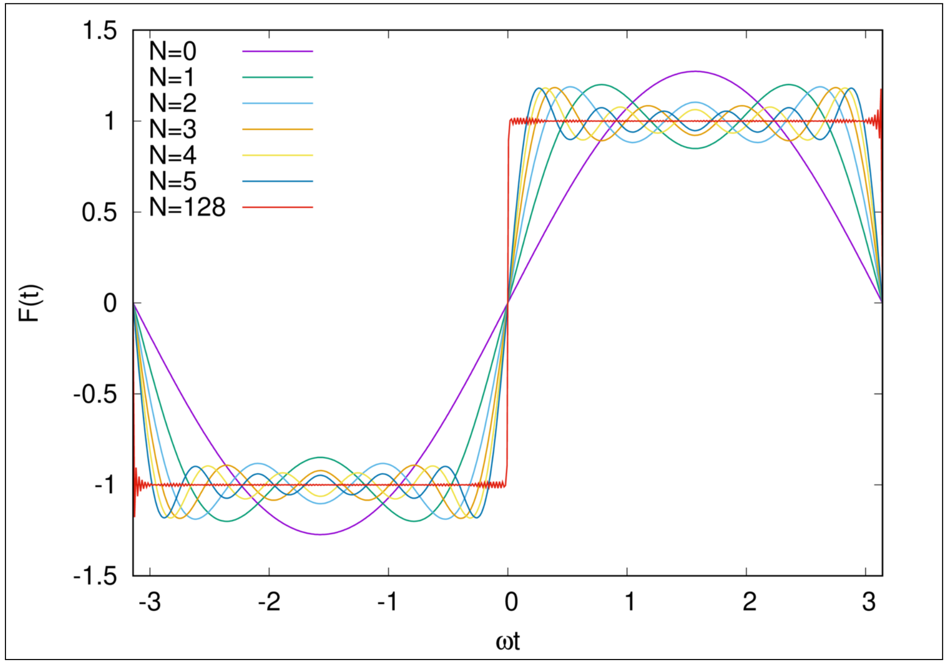
(11)

$$= \frac{\omega}{\pi}\left\{ \frac{\cos(n\omega t')}{n\omega}\Big|_{-\pi/\omega}^{0} + \left(-\frac{\cos(n\omega t')}{n\omega}\Big|_{0}^{\pi/\omega}\right)\right\}$$

$$= -\frac{\omega}{\pi}\left\{2\frac{\cos(n\pi)}{n\omega} - 2\frac{\cos(0)}{n\omega}\right\}$$

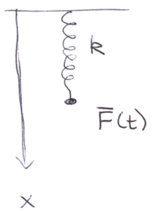
$$= \frac{2}{\pi n}\left\{1-\cos(n\pi)\right\} = \begin{cases} \frac{4}{n\pi} & n \in \text{oddatala} \\ 0 & n \text{ jöfn} \end{cases}$$

$$\to F(t) = \frac{4}{\pi}\sum_{n=0,1,2,\dots}^{\infty} \frac{\sin((2n+1)\omega t)}{(2n+1)}$$



03-09

(13)



$$\bar{F}(t) = \Theta(t)\Theta(t_0-t)F\hat{x}$$

$0 < t < t_0$

$$m\ddot{x} = -k(x-x_0) + F$$

$t > t_0$

$$m\ddot{x} = -k(x-x_0)$$

Sýna að eftir t_0 sé lausnin

$$x - x_0 = \frac{F}{k}\left\{\cos(\omega_0(t-t_0)) - \cos(\omega_0 t)\right\}$$

með $\omega_0^2 = \frac{k}{m}$

(14)

Skipta um breytu $y = x - x_0$

$\to$ Hreyfijafnan verður
$$\begin{cases} m\ddot{y} = -ky + F & \text{þ. } 0<t<t_0 \quad (1) \\ m\ddot{y} = -ky & \text{þ. } t>t_0 \quad (2) \end{cases}$$

Lausn (2) er

$$y(t) = Ae^{i\omega t} + Be^{-i\omega t}, \quad \omega = \sqrt{\frac{k}{m}} \qquad t > t_0$$

Lausn (1) er

$$y(t) = Ce^{i\omega t} + De^{-i\omega t} + \frac{F}{k} \qquad 0 < t < t_0$$

$\to$ þar sem síðasti liðurinn er sérlausn á hliðruðu jöfnunni (1)

Veljum upphafsskilyrði

$$y(0) = 0$$
$$\dot{y}(0) = 0$$

$\Leftarrow$ F(t) kemur kerfinu í gang....

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$$\dot{y}(0)=0 \;\to\; i\omega C - i\omega D = 0$$

$$\left.\begin{array}{l} y(0)=0 \;\to\; C + D + \frac{F}{k} = 0 \\ \dot{y}(0)=0 \;\to\; i\omega C - i\omega D = 0 \end{array}\right\} \to C = D = -\frac{F}{2k}$$

lausnin er samfeld í t_0, bæði y og $\dot{y}$

$y(t_0^-) = y(t_0^+)$

$$-\frac{F}{2k}e^{i\omega t_0} - \frac{F}{2k}e^{-i\omega t_0} + \frac{F}{k} = Ae^{i\omega t_0} + Be^{-i\omega t_0}$$

$\dot{y}(t_0^-) = \dot{y}(t_0^+)$

$$-\frac{F}{2k}i\omega e^{i\omega t_0} + \frac{F}{2k}i\omega e^{-i\omega t_0} = i\omega A e^{i\omega t_0} - i\omega B e^{-i\omega t_0}$$

16

$$\frac{F}{k}\left\{1 - \cos(\omega t_0)\right\} = Ae^{i\omega t_0} + Be^{-i\omega t_0}$$

$$-\frac{F}{k}\sin(\omega t_0) = Ae^{i\omega t_0} - Be^{-i\omega t_0}$$

$$\begin{pmatrix} e^{i\omega t_0} & e^{-i\omega t_0} \\ e^{i\omega t_0} & -e^{-i\omega t_0} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \frac{F}{k}\begin{pmatrix} 1-\cos(\omega t_0) \\ -\sin(\omega t_0) \end{pmatrix}$$

$$\to \boxed{\begin{array}{l} A = \frac{F}{2k}e^{-i\omega t_0}\left(1 - e^{+i\omega t_0}\right) \\ B = \frac{F}{2k}e^{+i\omega t_0}\left(1 - e^{-i\omega t_0}\right) \end{array}}$$

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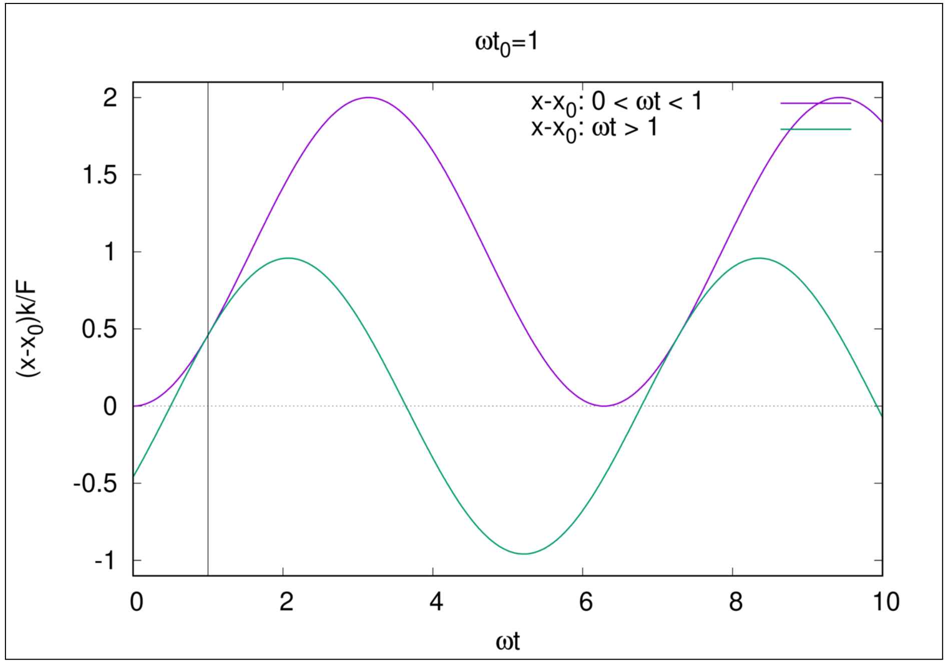
Lausnin

$$y(t) = \frac{F}{k}\left\{1 - \frac{e^{i\omega t}}{2} - \frac{e^{-i\omega t}}{2}\right\}$$

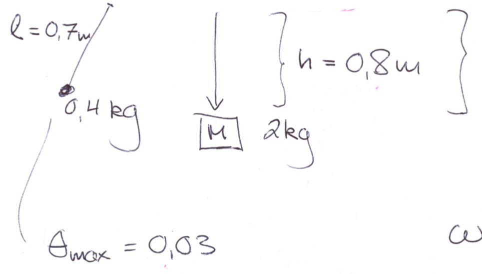
$$= \frac{F}{k}\left\{1 - \cos(\omega t)\right\}, \quad 0 < t < t_0$$

$$y(t) = \frac{F}{2k}\left\{e^{i\omega(t-t_0)}\left(1 - e^{i\omega t_0}\right) + e^{-i\omega(t-t_0)}\left(1 - e^{-i\omega t_0}\right)\right\}$$

$$= \frac{F}{k}\left\{\cos(\omega(t-t_0)) - \cos(\omega t)\right\}, \quad t > t_0$$



3-45 Finna Q fyrir klukku, $Q = \frac{\omega_R}{2\beta}$



$\theta_{max} = 0{,}03$ $\qquad \omega_R^2 = \omega_0^2 - 2\beta^2, \quad \omega_0 = \sqrt{\frac{g}{l}}$

Fyrir veika deyfingu er lausnin fyrir sveifluna

$\theta(t) = \theta_{max} \cdot e^{-\beta t}$ þ.s. β er dofnunarstuðullinn úr hreyfijöfnunni

$$\ddot{\theta} + 2\sqrt{g/l}\,\dot{\theta} + \frac{g}{l}\theta = 0$$

hér vantar í reynd lið sem bætir við orku

Þurfum að meta max gildi á β sem er þannig að orkan út úr kerfinu jafnist á við stöðuorku lóðs

$$E_{pot} = Mgh$$

sem nægir í 7d t.a. halda því gangandi, $\tau = 7d$

Upphafsorka pendúls

$$E_p(0) = \frac{1}{2} mgl\theta^2$$

Lota hans er $T = 2\pi\sqrt{\frac{l}{g}}$

$$\to E_p(T) = \frac{1}{2} mgl\,\theta^2(T) = \frac{1}{2} mgl\,\theta_{max}^2 e^{-2\beta T}$$

því er orkan týnd í einni sveiflu

$$E_p(T) - E_p(0) = \frac{1}{2} mgl\,\theta_{max}^2 \left\{1 - e^{-2\beta T}\right\}$$

$$E_p(T) - E_p(0) \simeq mgl\,\theta_{max}^2 \beta T$$

þess vegna fyrir tímabilið τ

$$mgl\,\theta_{max}^2 \beta T\left(\frac{\tau}{T}\right) = mgl\,\theta_{max}^2 \beta \tau$$

sú orka verður að vera jöfn stöðuorku M

$$\to mgl\,\theta_{max}^2 \beta\tau = Mgh$$

$$\to \beta = \frac{Mh}{ml\,\theta_{max}^2 \tau}$$

$$Q = \frac{\omega_R}{2\beta} = \frac{\sqrt{\frac{g}{l} - 2\beta^2}}{2\beta} = \frac{1}{2}\sqrt{\frac{g}{l\beta^2} - 2}$$

$$= \frac{1}{2}\sqrt{\frac{g m^2 l\,\theta_{max}^4 \tau^2}{M^2 h^2} - 2} \simeq 178.3$$

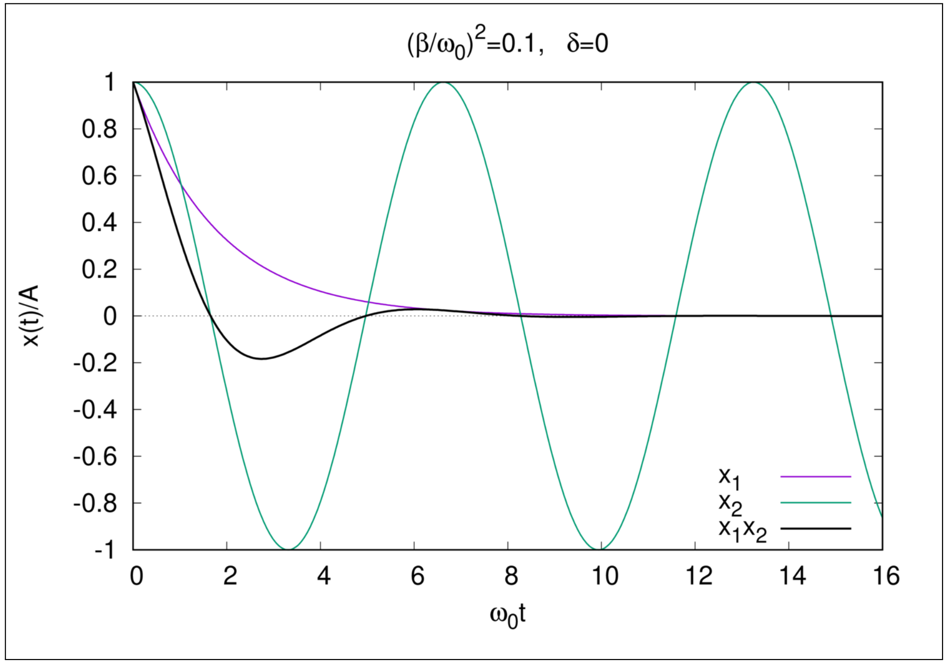
03-23 $\qquad x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$

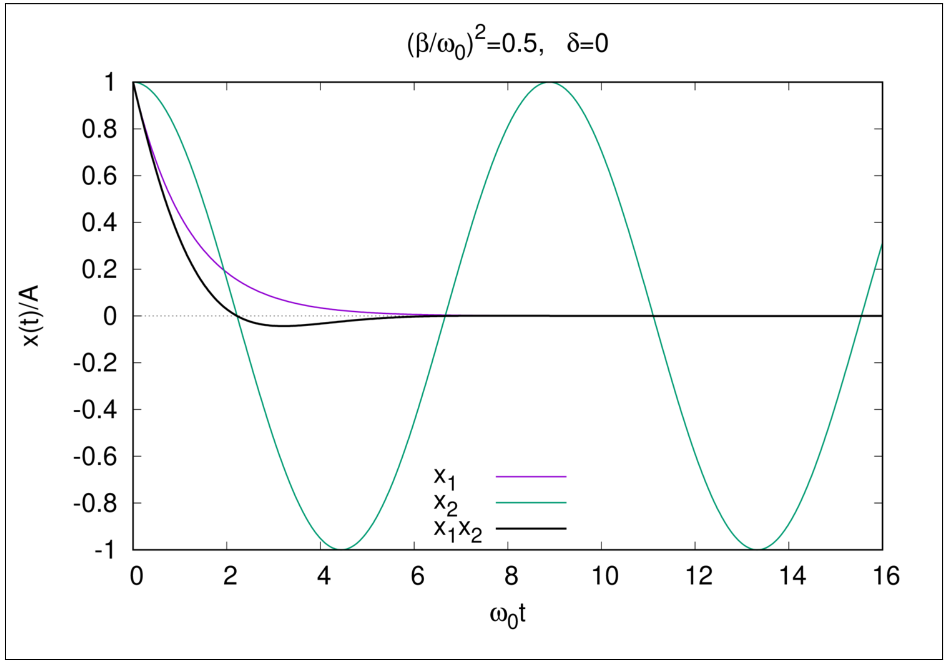
$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \omega_0\sqrt{1 - \frac{\beta^2}{\omega_0^2}}$$

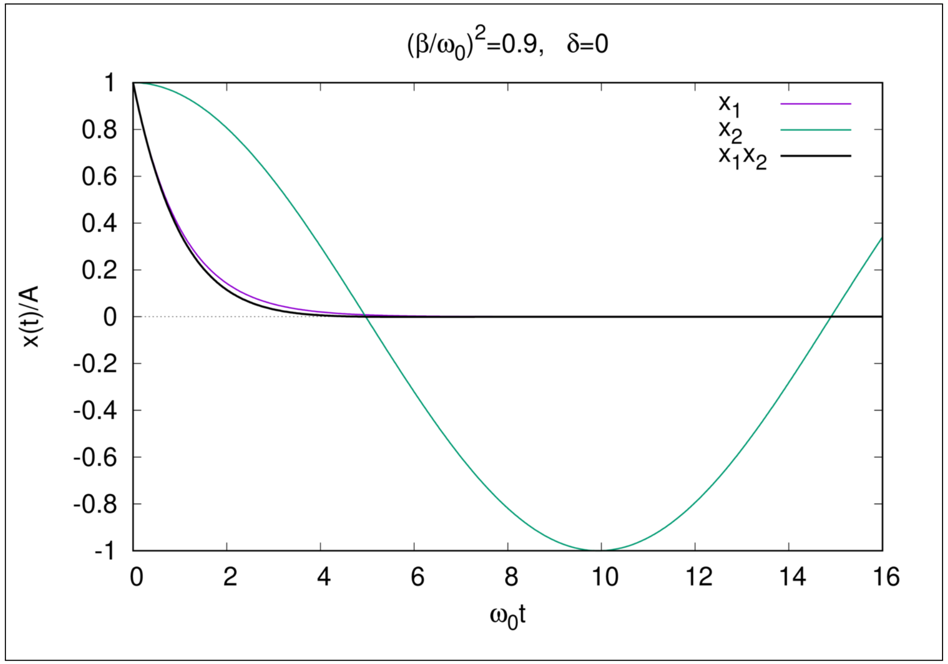
$\frac{\beta^2}{\omega_0^2} = 0.1,\ 0.5,\ 0.9$ $\quad | \quad$ $\delta = 0,\ \frac{\pi}{2},\ \pi$

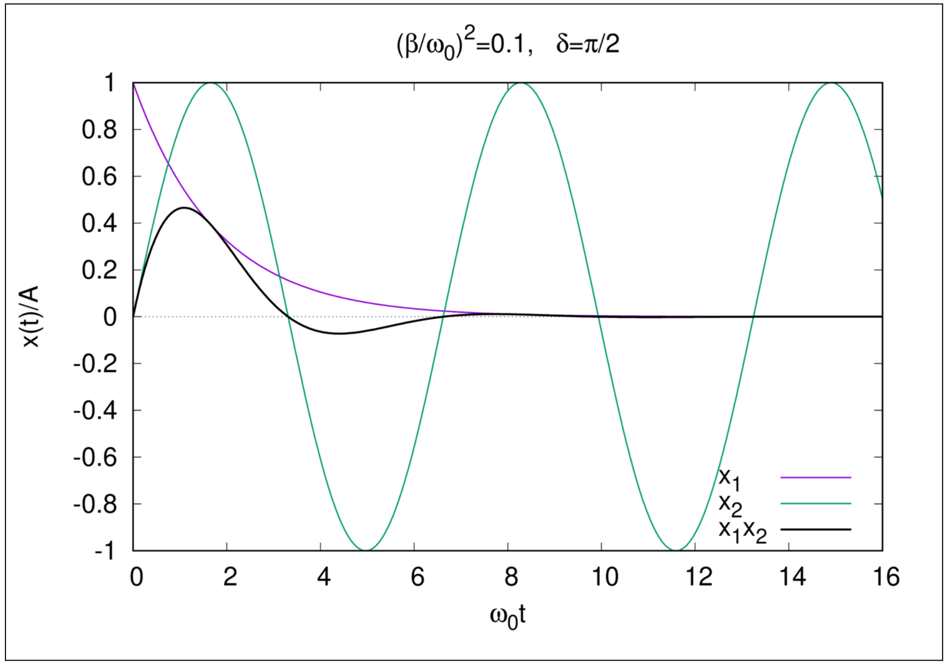
$$\frac{x(t)}{A} = \underbrace{\exp\left\{-\left(\frac{\beta}{\omega_0}\right)(t\omega_0)\right\}}_{x_1(t)} \underbrace{\cos\left\{(t\omega_0)\sqrt{1-\left(\frac{\beta}{\omega_0}\right)^2} - \delta\right\}}_{x_2(t)}$$

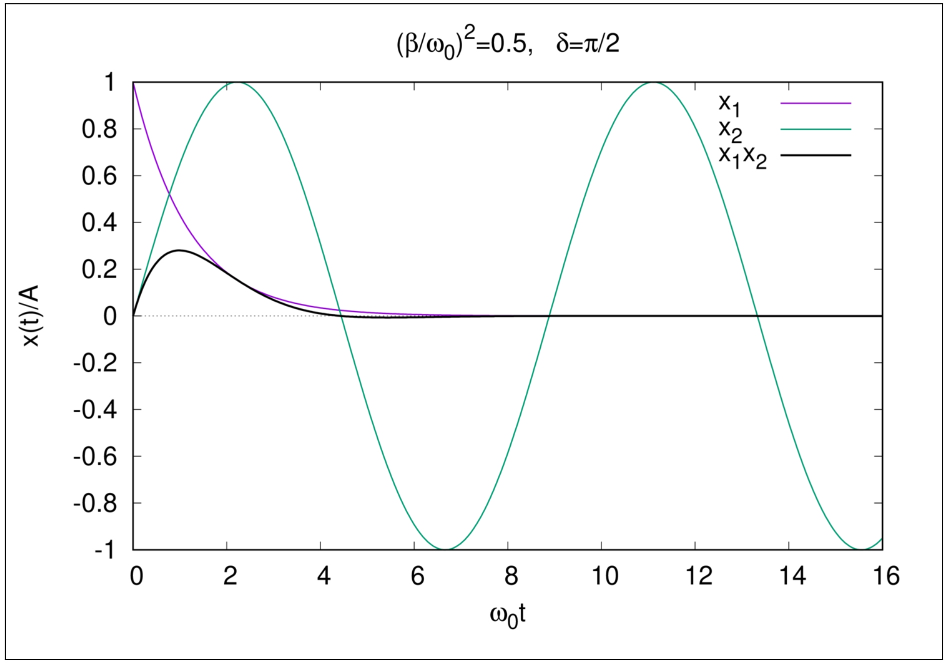
Gröfin fylgja

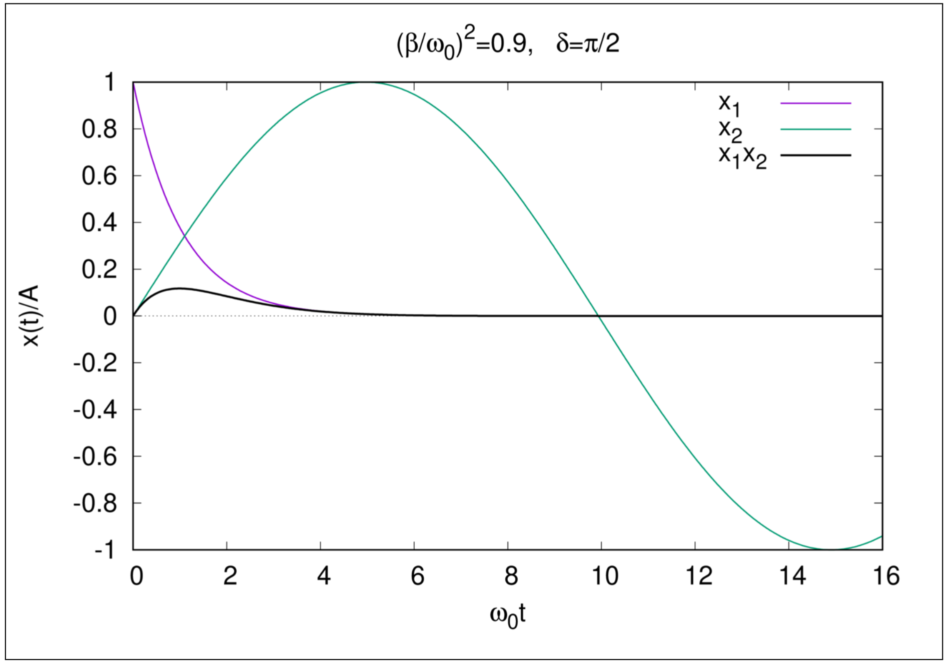

$(\beta/\omega_0)^2$=0.1, δ=0
x(t)/A
ω_0t
x_1
x_2
x_1x_2

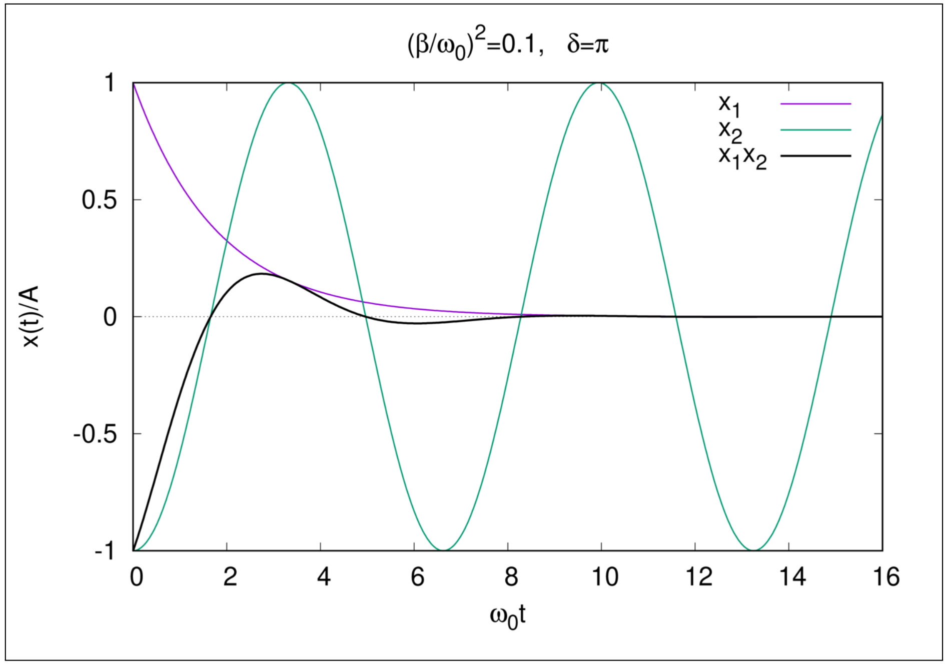

$(\beta/\omega_0)^2$=0.5, δ=0
x(t)/A
ω_0t
x_1
x_2
x_1x_2

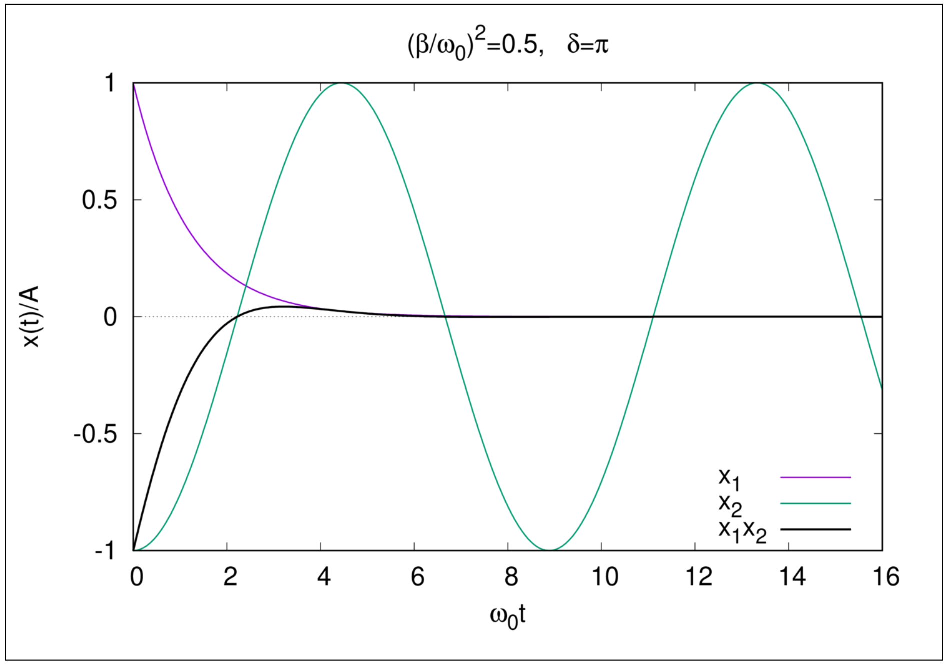

$(\beta/\omega_0)^2$=0.9, δ=0
x(t)/A
ω_0t
x_1
x_2
x_1x_2

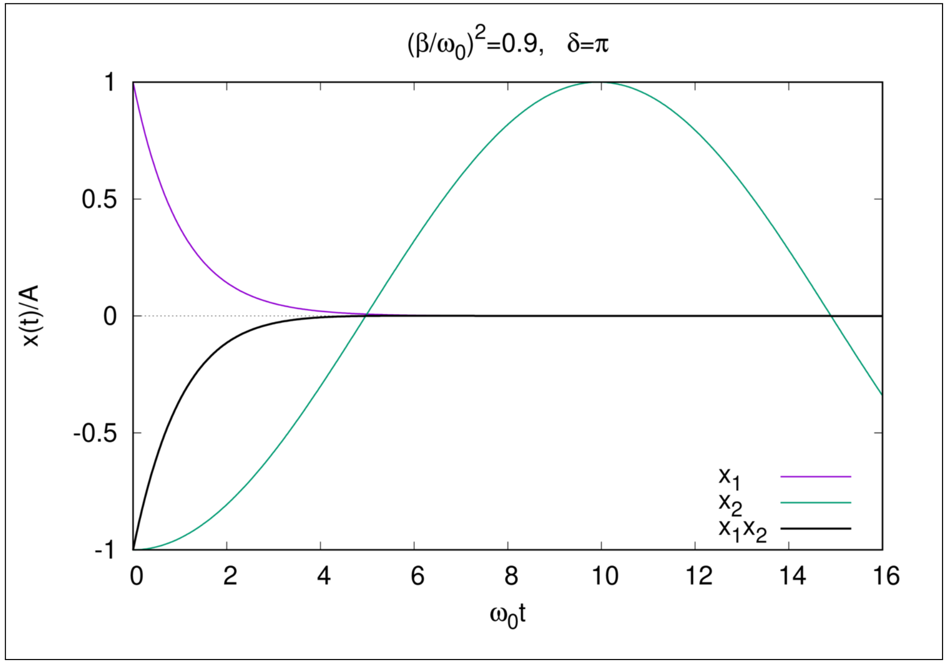

$(\beta/\omega_0)^2$=0.1, δ=π/2
x(t)/A
ω_0t
x_1
x_2
x_1x_2


(β/ω0)2=0.5, δ=π/2
x(t)/A
ω0t
x1
x2
x1x2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0 2 4 6 8 10 12 14 16


(β/ω0)2=0.9, δ=π/2
x(t)/A
ω0t
x1
x2
x1x2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0 2 4 6 8 10 12 14 16


(β/ω0)2=0.1, δ=π
x(t)/A
ω0t
x1
x2
x1x2
1
0.5
0
-0.5
-1
0 2 4 6 8 10 12 14 16


(β/ω0)2=0.5, δ=π
x(t)/A
ω0t
x1
x2
x1x2
1
0.5
0
-0.5
-1
0 2 4 6 8 10 12 14 16



(1)

04-06 Einvíður pendúll með massa m og lengd l

Stöðuorka, mætti: $U = mgl\{1-\cos\theta\}$

Hreyfiorka: $T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$

Mættið $U(\theta)$ er lotubundið eins og sést á mynd á síðu (3).

Könnum feril í fasarúminu $\{\theta, \dot{\theta}\}$ fyrir orku $2mgl$ og sitthvoru megin við. Þ.e. þegar hreyfingin er að verða hringur í $\{t, \theta\}$ sléttunni

$E = T + U$, ekkert tap

$$= \frac{1}{2}ml^2\dot{\theta}^2 + mgl\{1-\cos\theta\}$$

(2)

$$\rightarrow \frac{1}{2}ml^2\dot{\theta}^2 = E - mgl\{1-\cos\theta\}$$

$$\rightarrow \dot{\theta}^2 = \frac{2}{ml^2}\left[E - mgl\{1-\cos\theta\}\right]$$

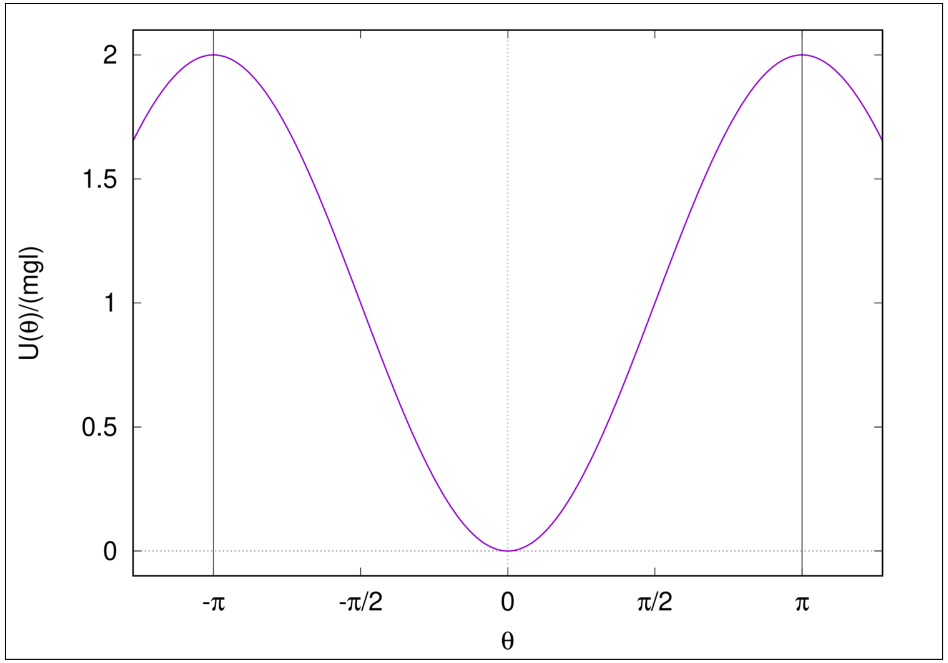
$$\rightarrow \dot{\theta} = \sqrt{\frac{2}{ml^2}}\left[E - mgl\{1-\cos\theta\}\right]^{1/2}$$

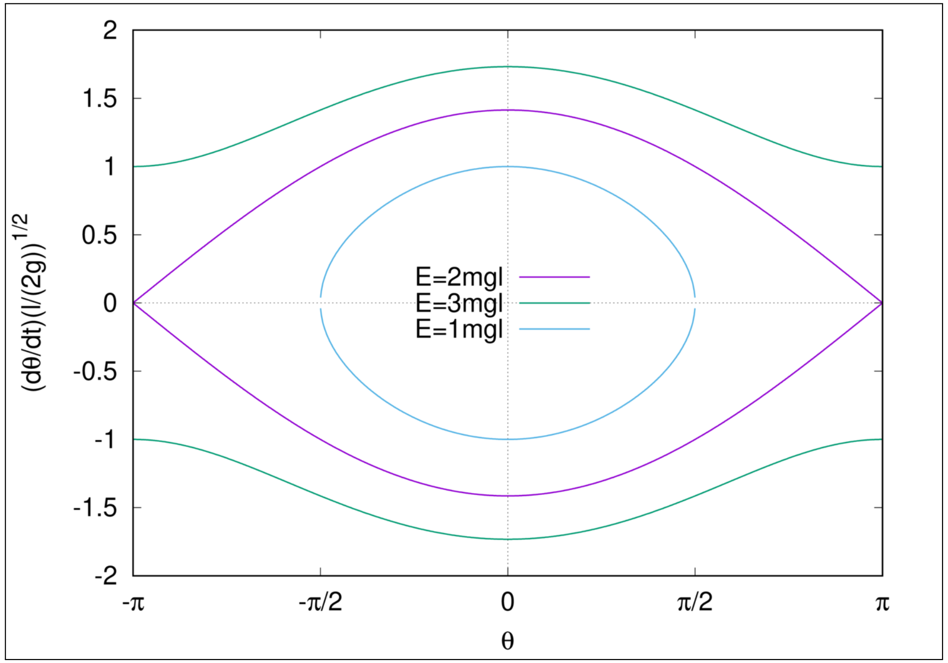
munum eftir tveimur rótum hér

Fyrir graf: $\dot{\theta} = \sqrt{\frac{2g}{l}}\left[\frac{E}{mgl} - (1-\cos\theta)\right]^{1/2}$ sjá mynd á síðu (4)

$$\rightarrow \dot{\theta}\sqrt{\frac{l}{2g}} = \left[\frac{E}{mgl} - (1-\cos\theta)\right]^{1/2}$$

Eðlilegar víddarlausar breytur fyrir graf





04-09

6

Taplaus hreyfing í krafti

$$F(x) = \begin{cases} -kx & |x|<a \\ -(k+\delta)x + \delta a & |x|>a \end{cases}$$

Kraftinum má lýsa með mættinu $\left(F = -\frac{d}{dx} U \right)$

$$U(x) = \begin{cases} \frac{kx^2}{2} & |x|<a \\ \frac{(k+\delta)x^2}{2} - \delta a x & |x|>a \end{cases}$$

skölum fyrir graf

$$= \begin{cases} \frac{ka^2}{2}\left(\frac{x}{a}\right)^2 & \left|\frac{x}{a}\right|<1 \\ \frac{(k+\delta)a^2}{2}\left(\frac{x}{a}\right)^2 - \delta a^2\left(\frac{x}{a}\right) & \left|\frac{x}{a}\right|>1 \end{cases}$$

8

$$E = T+U, \qquad T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

V er víddarlaust

$$T = E-U \;\rightarrow\; \frac{1}{2}mv^2 = E - U(x)$$

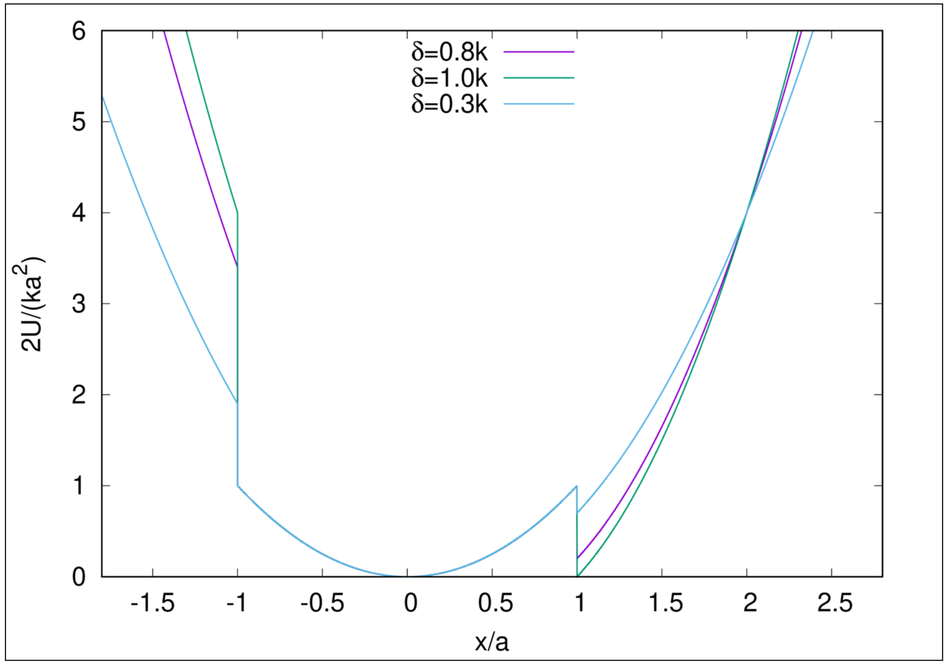
$$v = \sqrt{\frac{2}{m}}\left(E - U(x)\right)^{1/2} = \sqrt{\frac{2}{m}}\left(E - \frac{ka^2}{2}V(x)\right)^{1/2}$$

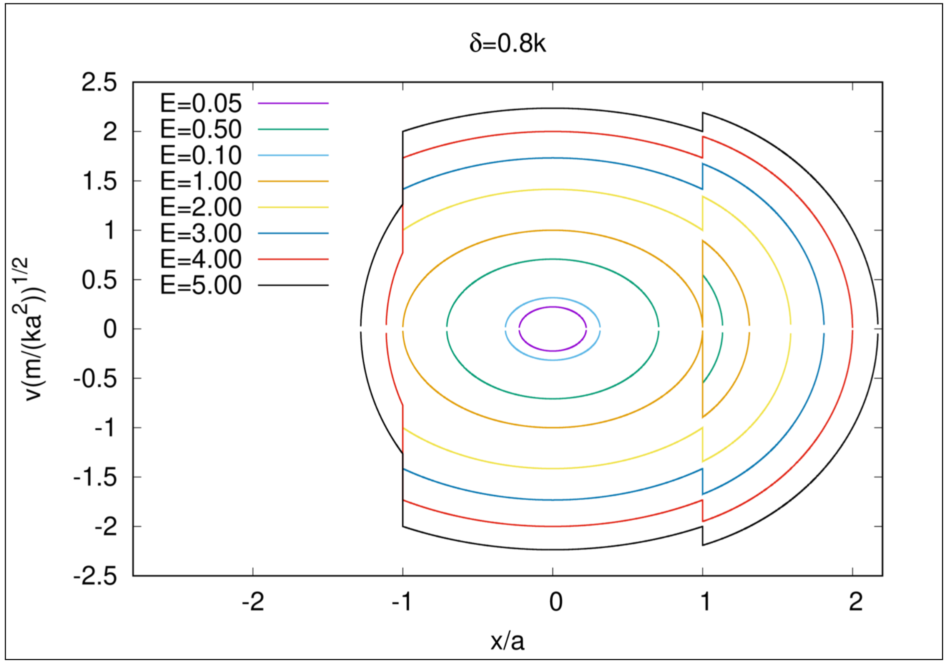
$$v = \sqrt{\frac{2}{m}\frac{ka^2}{2}}\left(\frac{2E}{ka^2} - V(x)\right)^{1/2}$$

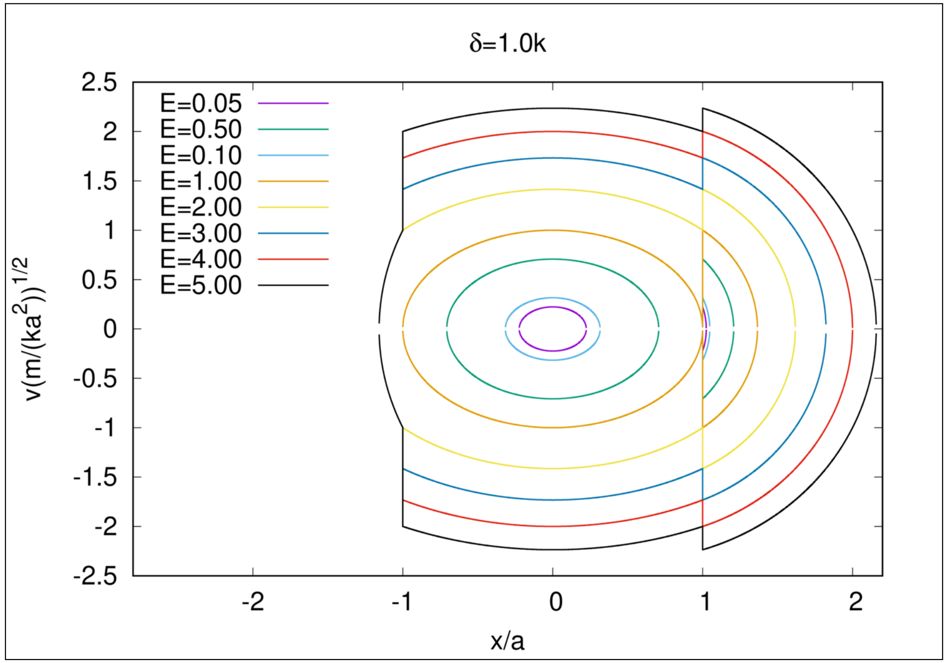
$$= \sqrt{\frac{ka^2}{m}}\left(\frac{2E}{ka^2} - V(x)\right)^{1/2}$$

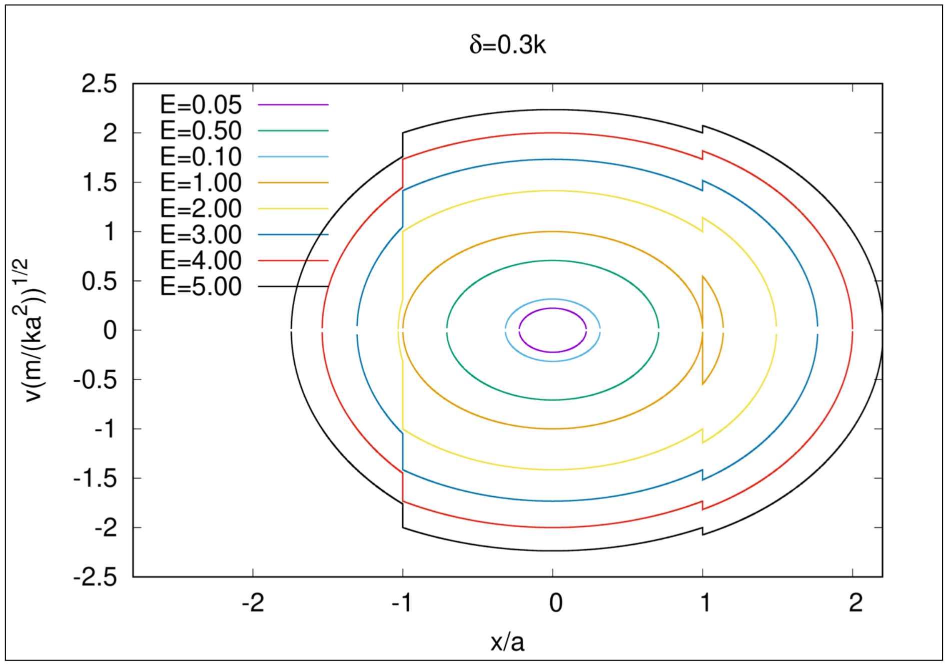
skölum orku notað á gröfunum í forsíðunni

Gröfin eru á næstu 3 síðum









04-03

$$U(x) = -\left(\frac{\lambda}{3}\right) x^3$$

Eins og í dæminu á undan er

$$v = \sqrt{\frac{2}{m}}\left(E - U(x)\right)^{1/2}$$

$$= \sqrt{\frac{2}{m}}\left(E + \frac{\lambda}{3}x^3\right)^{1/2}$$

stöðlum

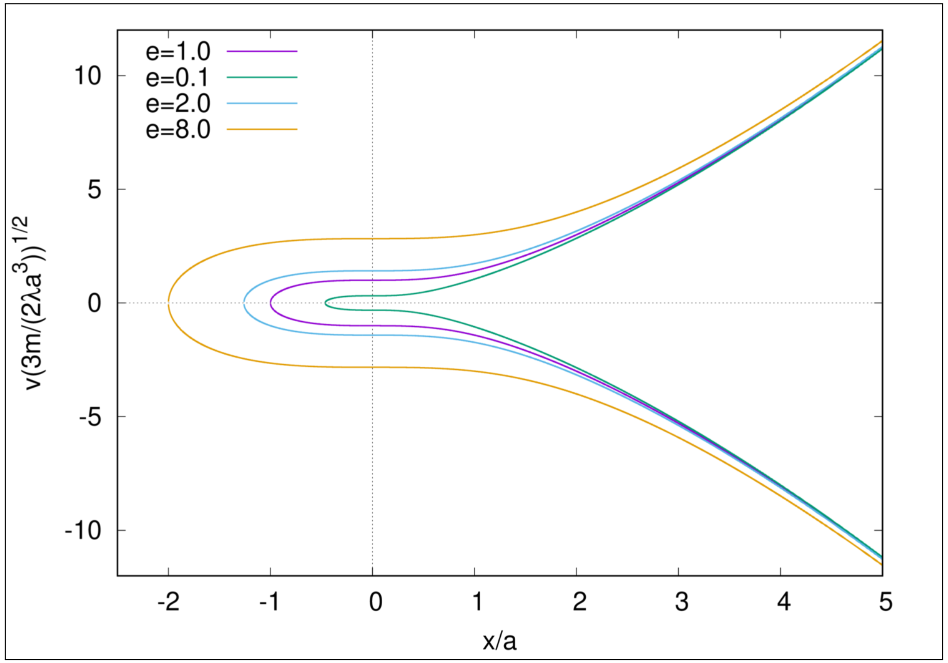
$$v = \sqrt{\frac{2}{m}}\left(E + \frac{\lambda a^3}{3}\left(\frac{x}{a}\right)^3\right)^{1/2}$$

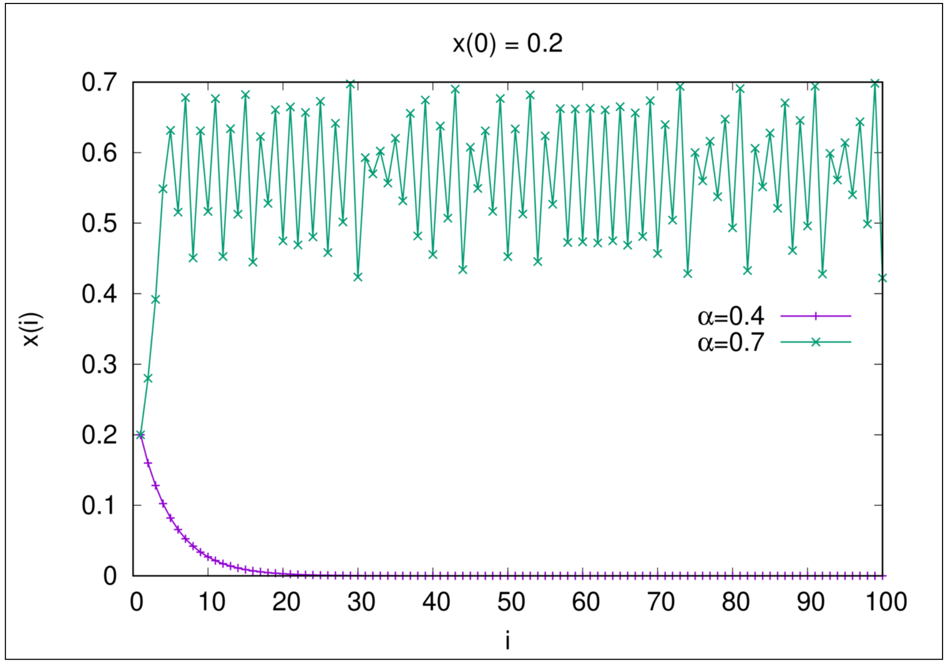
$$= \sqrt{\frac{2\lambda a^3}{3m}}\left(\frac{3E}{\lambda a^3} + \left(\frac{x}{a}\right)^3\right)^{1/2}$$

$$v\sqrt{\frac{3m}{2\lambda a^3}} = \left(\frac{3E}{\lambda a^3} + \left(\frac{x}{a}\right)^3\right)^{1/2}$$

"E" á grafi

Sjá næstu síðu





15

04-17 Tjald-vörpunin

$$x_{n+1} = 2\alpha x_n \qquad 0 < x < 1/2 \qquad\qquad 0 < \alpha < 1$$

$$x_{n+1} = 2\alpha(1 - x_n) \qquad \tfrac{1}{2} < x < 1$$

Kanna fyrir $\alpha = 0,4$ og $0,7$ með $x_1 = 0,2$

Sjá graf á næstu síðu

18

04-04 Jafna Rayleigh er

$$\ddot{x} - (a - b\dot{x}^2)\dot{x} + \omega_0^2 x = 0 \qquad (*)$$

Sýna að breytuskiptin $\boxed{y = y_0\sqrt{3\frac{b}{a}}\,\dot{x}}$ leiði til jöfnu van der Pol

$$\ddot{y} - \frac{a}{y_0^2}(y_0^2 - y^2)\dot{y} + \omega_0^2 y = 0$$

Breytuskiptin benda til þess að best sé að diffra (*)

$$\dddot{x} + 2b\dot{x}\ddot{x}\ddot{x} - (a - b\dot{x}^2)\ddot{x} + \omega_0^2\dot{x} = 0$$

$$\rightarrow \quad \dddot{x} - \{a - 3b\dot{x}^2\}\ddot{x} + \omega_0^2\dot{x} = 0$$

$$Y = Y_0\sqrt{\frac{3b}{a}}\,\dot{x} \longrightarrow \dot{x} = \sqrt{\frac{a}{3b}}\,\frac{Y}{Y_0}$$

$$\longrightarrow \ddot{x} = \sqrt{\frac{a}{3b}}\,\frac{\dot{Y}}{Y_0} \qquad \dddot{x} = \sqrt{\frac{a}{3b}}\,\frac{\ddot{Y}}{Y_0}$$

Innsetning

$$\sqrt{\frac{a}{3b}}\left\{\frac{\ddot{Y}}{Y_0} - \left[a - 3b\frac{a}{3b}\frac{Y^2}{Y_0^2}\right]\frac{\dot{Y}}{Y_0} + \omega_0^2\frac{Y}{Y_0}\right\} = 0$$

$$\longrightarrow \ddot{Y} - \frac{a}{Y_0^2}\left\{Y_0^2 - Y^2\right\}\dot{Y} + \omega_0^2 Y = 0$$

Jafna van der Pols

04-22

$$\frac{dx}{dt} = y$$

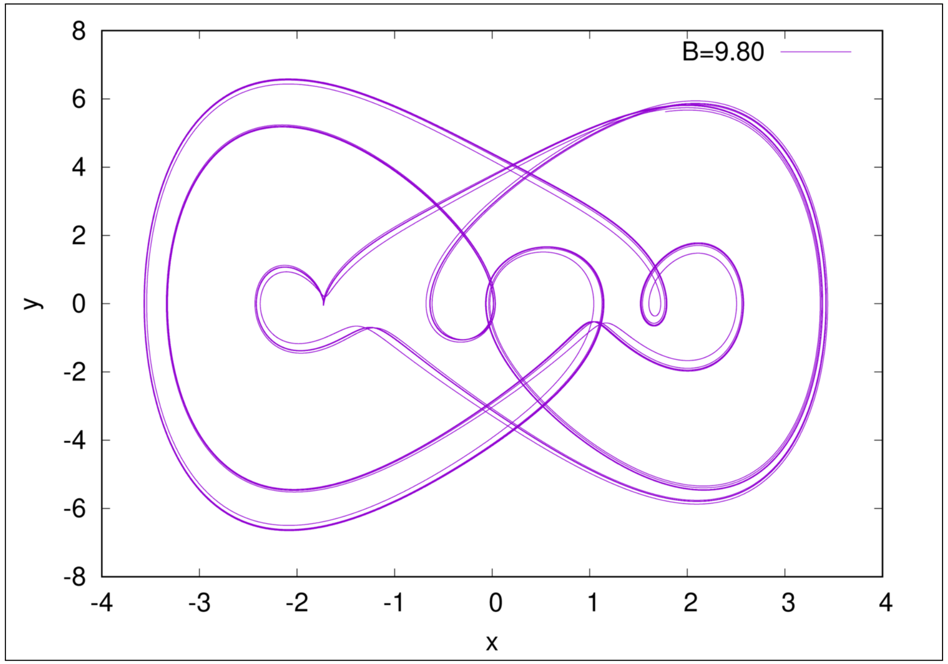
$$\frac{dy}{dt} = -ky - x^3 + B\cos t$$

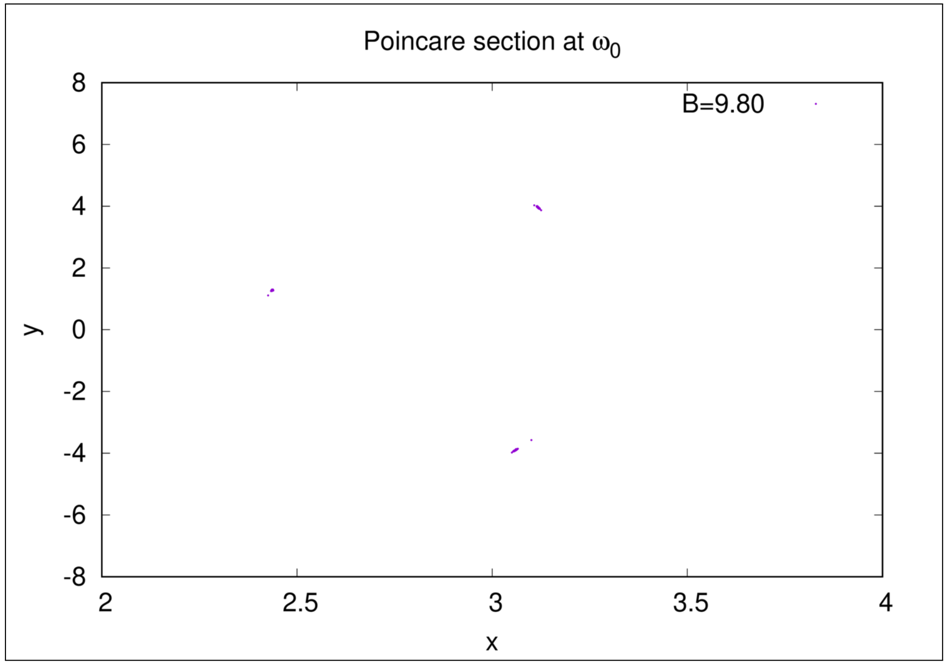
Poincaré snið : $k = 0{,}1$ $\quad 9.8 \leq B \leq 13.4$

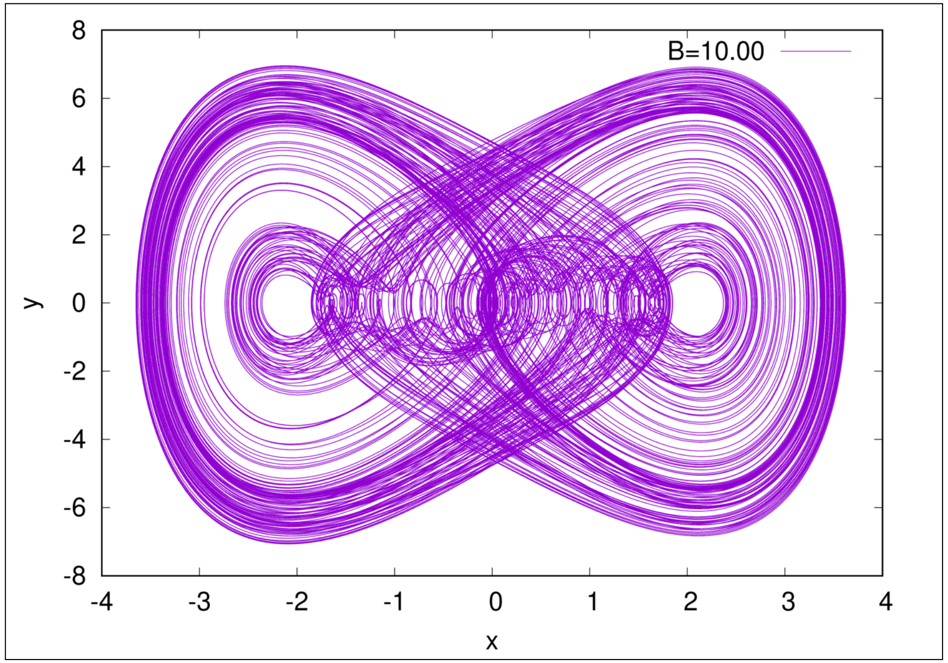
Eg bý til fasarit fyrir 4 gildi á B, en kanna ikki alt sniðið. Eg nýta 1 000 000 punkt í tíma og sleppi 100000 fyrstu punktunum. Í fasaritinum er $\Delta t = 0{,}001$, en í sniðinum nýta eg $t_n = 2\pi n$ sum samsvarar $\omega_0 = 1$ her.

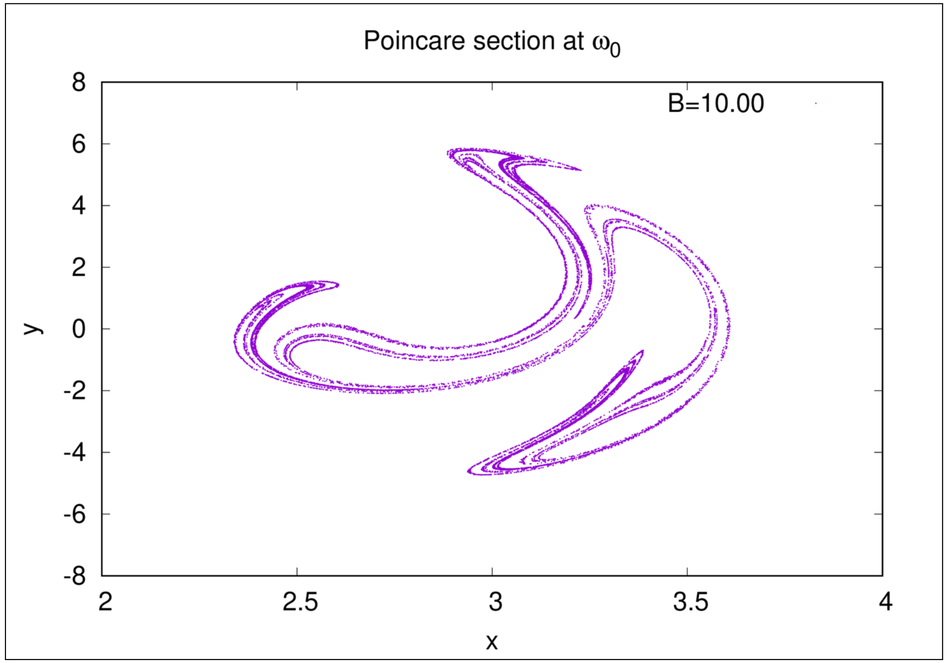
Hver koyrsla í gfortran tekur ~ 2s.

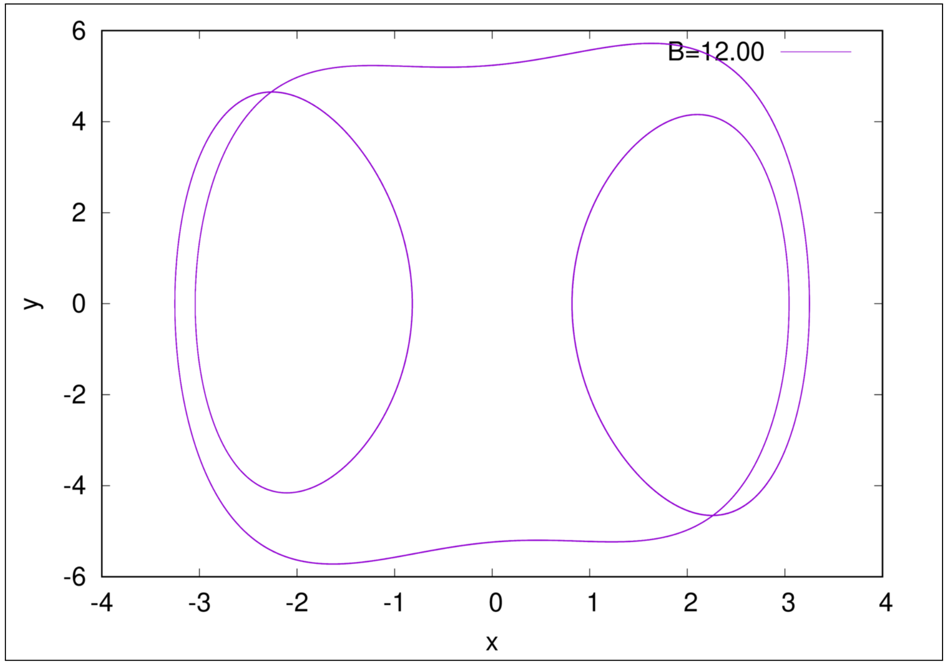
Upphafspunktur $(x,y) = (0,0)$

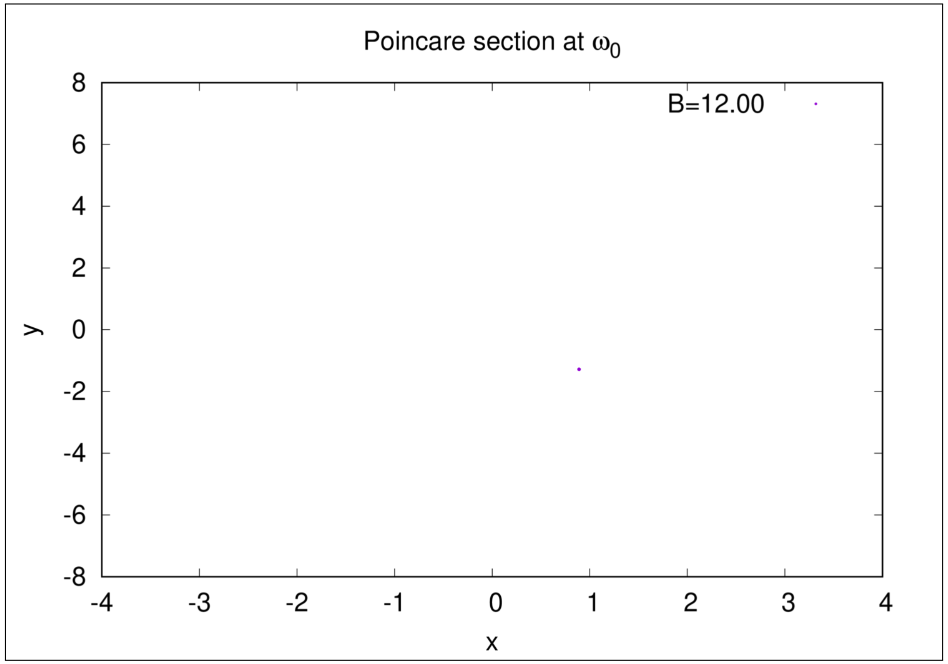


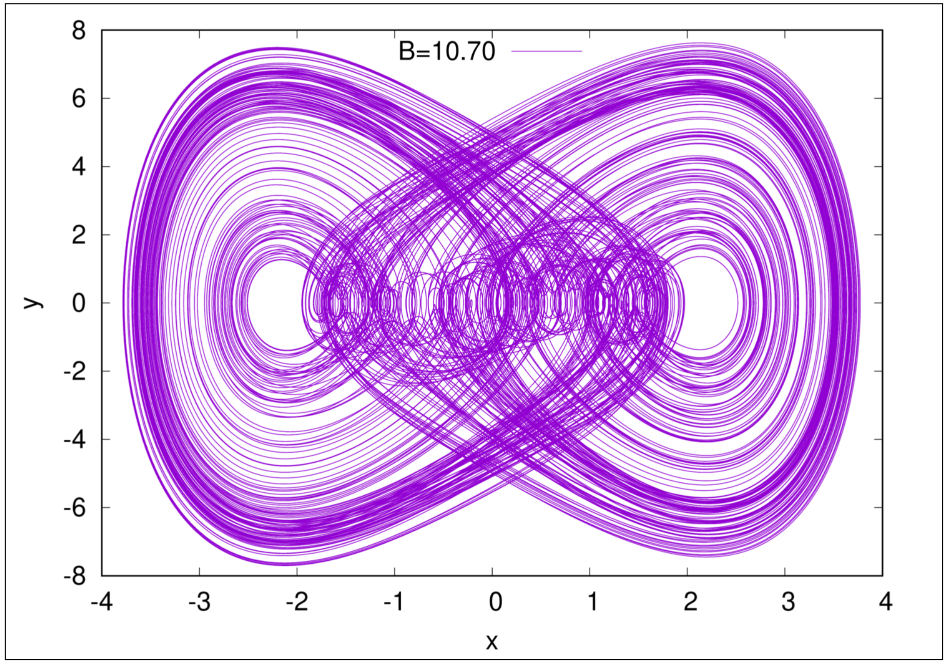


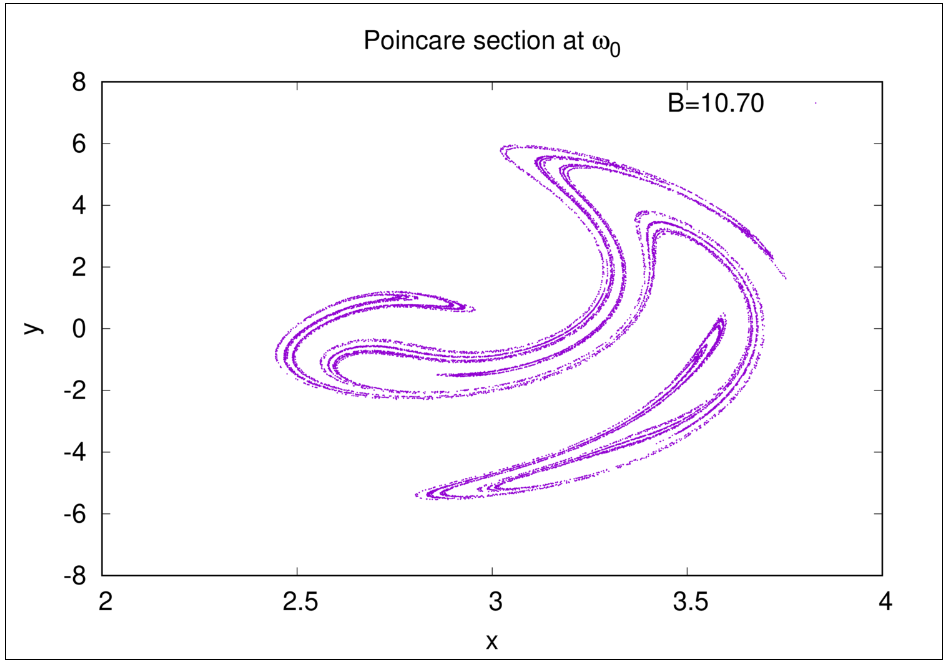

B=10.00
y
x


Poincare section at ω_0
B=10.00
y
x


B=12.00
y
x


Poincare section at ω_0
B=12.00
y
x

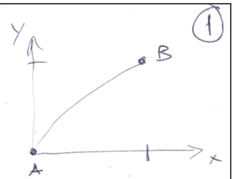




6-1

1

Milli punkta (0,0) og (1,1) í sléttu línka



$$y(\alpha,x) = x + \alpha \cdot \mathrm{Sin}\left[\pi(1-x)\right]$$

til að sjá að $y(0,x) = x$ sé Stysti ferillinn (sjá mynd á næstu síðu).

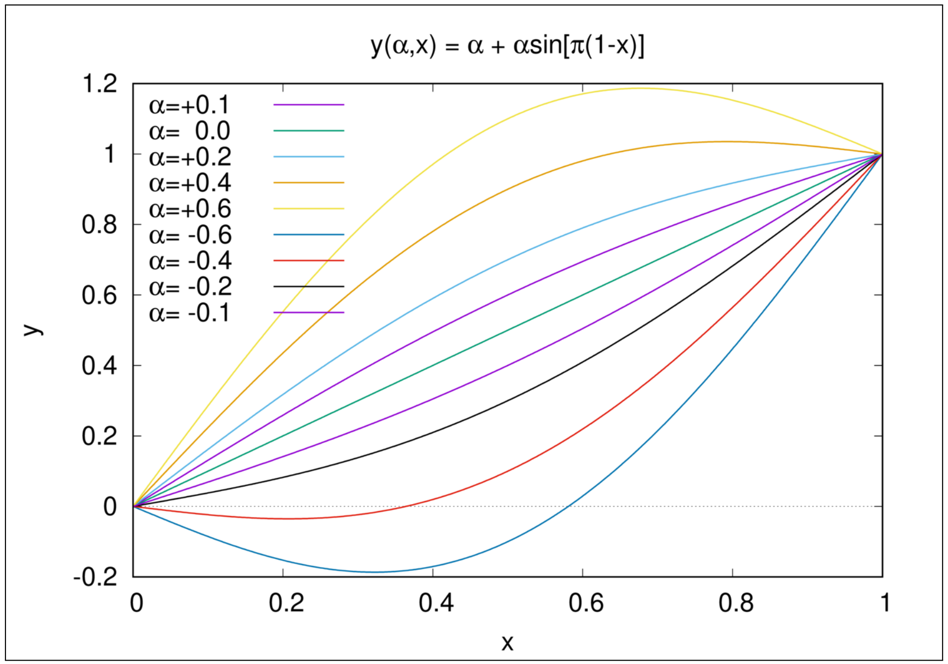
Lengd stöðarinnar er

$$S = \int_A^B \sqrt{dx^2 + dy^2}$$

$$\frac{dy(\alpha,x)}{dx} = 1 - \alpha\pi \, \mathrm{Cos}\left[\pi(1-x)\right]$$

sem við útfærum með $y(\alpha,x)$ sem

$$S(\alpha) = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$



$$S(\alpha) = \int_0^1 dx \sqrt{1 + \{1 - \alpha\pi \cos[\pi(1-x)]\}^2}$$ gegnum breytuskipti $u = \pi(1-x)$

$$= \frac{1}{\pi}\int_0^\pi du \sqrt{1 + \{1 - \alpha\pi \cos(u)\}^2}$$

$$= \frac{\sqrt{2}}{\pi}\int_0^\pi du \sqrt{1 - \alpha\pi\cos(u) + \frac{(\alpha\pi)^2}{2}\cos^2(u)}$$

Elliptískt heildi gegnum röð fyrir $\alpha \ll 1$

$$\approx \frac{\sqrt{2}}{\pi}\int_0^\pi du \left\{1 - \frac{1}{2}\left(\alpha\pi\cos(u) - \frac{(\alpha\pi)^2}{2}\cos^2(u)\right) - \frac{1}{8}\left(\alpha\pi\cos(u) - \frac{(\alpha\pi)^2}{2}\cos^2(u)\right)^2 + \cdots\right\}$$

nú þarf að tína saman liði með minnst veldi $(\alpha\pi)^2$ og heilda

$$S(\alpha) \simeq \sqrt{2} + \frac{\sqrt{2}}{\pi}\int_0^\pi du \left\{\underbrace{-\frac{\alpha\pi}{2}\cos(u)}_{\text{gefur } 0} + \frac{(\alpha\pi)^2}{4}\cos^2(u) - \frac{(\alpha\pi)^2}{8}\cos^2(u)\right\} + \cdots$$

$$\approx \sqrt{2} + \frac{\sqrt{2}}{\pi}\int_0^\pi du \frac{(\alpha\pi)^2}{8}\cos^2(u) + \cdots \approx \sqrt{2} + \frac{\sqrt{2}}{\pi}\frac{(\alpha\pi)^2}{8}\frac{\pi}{2}$$

$$= \sqrt{2} + \frac{\sqrt{2}}{16}(\pi\alpha)^2 + \cdots$$

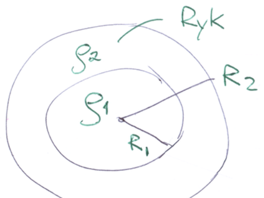
$$\frac{\partial S(\alpha)}{\partial \alpha} = \frac{\sqrt{2}\pi^2\alpha}{8} \longrightarrow 0 \text{ þegar } \alpha = 0$$

→ jafna fyrir beinalínu

Athugið að hér finnst lággmark þ. $\alpha = 0$ og $\alpha \ll 1$
Við getum misst af öðrum lággmörkum fyrir stærri α...

5-13 Reikistjarna

Finna kraftinn á ryk ögn með massa m



Lögmál Gauß gildir

$$\oint \bar{g}\cdot d\bar{s} = -4\pi G M$$

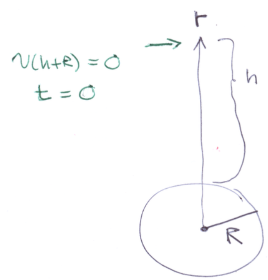
Ef $\rho = \rho(r)$, óháð θ og ϕ þá er $\bar{g}$ einsleitt $\bar{g} = g\hat{e}_r$
M gæti líka allt eins verið punktmassi, massinn utan r skiptir ekki máli

$$\rightarrow \bar{F}(r) = m\bar{g}(r) = -m\frac{GM(r)}{r^2}\hat{e}_r$$

$$M(r) = \underbrace{\frac{4\pi}{3}R_1^3\rho_1}_{\text{massi reikistjörnu}} + \underbrace{\frac{4\pi}{3}(r^3 - R_1^3)\rho_2}_{\text{massinn úr rykhjúpnum}}$$

$$\bar{F}(r) = -\frac{4\pi m}{3}G\left\{\frac{(\rho_1-\rho_2)R_1^3}{r^2} + \rho_2 r\right\}\hat{e}_r$$

05-05 Fall að jörð, sýna að fall um $\frac{h}{2}$ taki um það bil $\frac{9}{11}T$ þ.s. T er falltíminn



Hreyfijafnan er

$$m\ddot{r} = -G\frac{Mm}{r^2} \qquad (1)$$

Orkuvarðveisla

$$\frac{1}{2}m\dot{r}^2 - GM\frac{m}{r} = E = -GM\frac{m}{h+R} \qquad (2)$$

(7)

Endurritun (2)

$$\frac{1}{2}m\dot{r}^2 = GMm\underbrace{\left\{\frac{1}{r}-\frac{1}{R+h}\right\}}_{\geq 0} \qquad r \leq R+h$$

$$\rightarrow \quad \dot{r} = \frac{dr}{dt} = -\sqrt{2GM\left\{\frac{1}{r}-\frac{1}{R+h}\right\}}$$

hraðinn er neikvæður niður á við

$$\rightarrow \quad dt = -\frac{dr}{\sqrt{2GM\left\{\frac{1}{r}-\frac{1}{R+h}\right\}}}$$

heildum

$$\int_0^t dt' = -\int_{R+h}^{r}\frac{dr'}{\sqrt{2GM\left\{\frac{1}{r'}-\frac{1}{R+h}\right\}}}$$

(8)

$$t = -\frac{1}{\sqrt{GM2}}\int_{R+h}^{r} dr'\sqrt{\frac{r'(R+h)}{R+h-r'}}$$

Skiptum um breytu

$$y^2 = r' \quad\rightarrow\quad 2y\,dy = dr' \;;\rightarrow\; \sqrt{r'}\,dr' = 2y^2dy$$

$$t = -\sqrt{\frac{2}{GM}}\int_{\sqrt{R+h}}^{\sqrt{r}} y^2dy\frac{\sqrt{R+h}}{\sqrt{R+h-y^2}}$$

notum heildi úr viðbót E (E7)

$$= -\sqrt{R+h}\sqrt{\frac{2}{GM}}\left[-\frac{y}{2}\sqrt{R+h-y^2}+\frac{R+h}{2}\arcsin\left(\frac{y}{\sqrt{R+h}}\right)\right]_{\sqrt{R+h}}^{\sqrt{r}}$$

(9)

$$t = \frac{\sqrt{R+h}}{\sqrt{2GM}}\left\{\sqrt{r}\sqrt{R+h-r} - (R+h)\arcsin\left(\sqrt{\frac{r}{R+h}}\right) + (R+h)\frac{\pi}{2}\right\}$$

Fall úr $r = h+R$ í R

$$t(R) = \frac{\sqrt{R+h}}{\sqrt{2GM}}\left\{\sqrt{Rh} - (R+h)\arcsin\left(\sqrt{\frac{R}{R+h}}\right) + \frac{(R+h)\pi}{2}\right\}$$

Fall úr $r = h+R$ í $R+\frac{h}{2}$

$$t\left(R+\frac{h}{2}\right) = \frac{\sqrt{R+h}}{\sqrt{2GM}}\left\{\sqrt{R+\frac{h}{2}}\sqrt{\frac{h}{2}} - (R+h)\arcsin\left(\sqrt{\frac{R+\frac{h}{2}}{R+h}}\right) + \frac{(R+h)}{2}\pi\right\}$$

$h >> R$

$$t(R) = \frac{h^{3/2}\sqrt{1+\frac{R}{h}}}{\sqrt{2GM}}\left\{\sqrt{\frac{R}{h}} - \left(1+\frac{R}{h}\right)\arcsin\left(\sqrt{\frac{\frac{R}{h}}{1+\frac{R}{h}}}\right) + \frac{\left(1+\frac{R}{h}\right)\pi}{2}\right\}$$

(10)

$$t\left(R+\frac{h}{2}\right) = \frac{h^{3/2}\sqrt{1+\frac{R}{h}}}{\sqrt{2GM}}\left\{\frac{1}{2}\sqrt{1+\frac{2R}{h}} - \left(1+\frac{R}{h}\right)\arcsin\left(\sqrt{\frac{\frac{1}{2}+\frac{R}{h}}{1+\frac{R}{h}}}\right) + \frac{\left(1+\frac{R}{h}\right)\pi}{2}\right\}$$

$h >> R$, reynum lægstu nálgun

$$t(R) \simeq \frac{h^{3/2}}{\sqrt{2GM}}\frac{\pi}{2}, \qquad t\left(R+\frac{h}{2}\right) \simeq \frac{h^{3/2}}{\sqrt{2GM}}\left\{\frac{1}{2} - \arcsin\left(\frac{1}{2}\right) + \frac{\pi}{2}\right\}$$

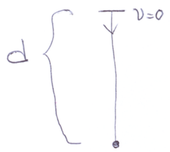
$$= \frac{h^{3/2}}{\sqrt{2GM}}\left\{\frac{1}{2} - \frac{\pi}{6} + \frac{\pi}{2}\right\}$$

$$\rightarrow \quad \frac{t\left(R+\frac{h}{2}\right)}{t(R)} = \frac{\left(\frac{3-\pi+3\pi}{6}\right)}{\pi/2}$$

$$= \frac{2}{6}\left(\frac{3+2\pi}{\pi}\right) = \frac{1}{3\pi}(3+2\pi)$$

$$= \frac{2\pi}{3\pi}\left(1+\frac{3}{2\pi}\right) = \frac{2}{3}\left(1+\frac{3}{2\pi}\right) = 0.985$$

05-04 (11)



$F = -\frac{mk^2}{r^3}$, sýna að $T_d = \frac{d^2}{k}$

$\bar F = -\bar\nabla U$ því getum við valið $U(r) = -\frac{mk^2}{2r^2}$

Miðlægt mætti og geymin kraftar

Orkuvarðveisla

$$E = \underbrace{\frac{m}{2}\dot r^2 - \frac{mk^2}{2r^2}}_{} = -\underbrace{\frac{mk^2}{2d^2}}_{}$$

Ef upphafsorkan er aðeins mættisorka, $v(d) = 0$
eða $v(t=0) = 0$

$$\frac{m\dot r^2}{2} = \frac{mk^2}{2}\left\{\frac{1}{r^2} - \frac{1}{d^2}\right\}$$

$$\left(\frac{dr}{dt}\right)^2 = k^2\left\{\frac{1}{r^2} - \frac{1}{d^2}\right\} \longrightarrow dt = \frac{dr}{k\sqrt{\frac{1}{r^2} - \frac{1}{d^2}}}$$

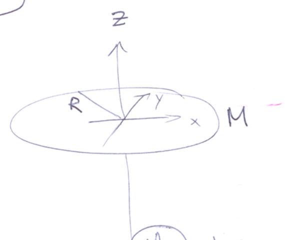
(12)

$$\int_0^t dt' = -\int_d^0 \frac{dr}{k\sqrt{\frac{1}{r^2} - \frac{1}{d^2}}} = -\frac{d}{k}\int_d^0 \frac{r\,dr}{\sqrt{d^2 - r^2}}$$

$$= +\frac{d}{k}\sqrt{d^2 - r^2}\,\Big|_d^0 = \frac{d^2}{k} = T_d$$

formerkið á rótinni er valið til að t vaxi þegar r minnkar

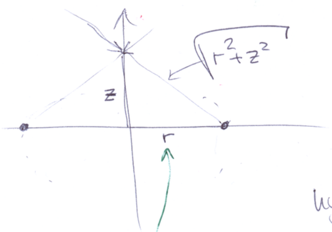
5-20 (13)



Reikna $\Phi(z)$ og $\bar g(z)$

á z-ás

Athugum mættið frá grönnum hring



Einfaldast er að skoða Ex. 5.4
í bók og setja saman Φ beint

$$\boxed{d\Phi = -G\frac{dM}{\sqrt{r^2+z^2}}}$$

hugsum $dM = \rho^{2D} dA = \rho^{2D} 2\pi r\,dr$

og $\rho^{2D} = \frac{M}{\pi R^2}$

← tvívíð massadreifing

(14)

$$dM = \left(\frac{M}{\pi R^2}\right) 2\pi r\,dr = 2M\frac{r\,dr}{R^2}$$ ← gott að sjá rétta vídd

$$\longrightarrow d\Phi = -G\frac{2M\,r\,dr}{R^2\sqrt{r^2+z^2}}$$

$$\Phi(z) = -\frac{G2M}{R^2}\int_0^R \frac{r\,dr}{\sqrt{r^2+z^2}} = -\frac{GM2}{R^2}\sqrt{r^2+z^2}\,\Big|_0^R$$

$$= -\frac{2GM}{R^2}\left\{\sqrt{R^2+z^2} - z\right\}$$

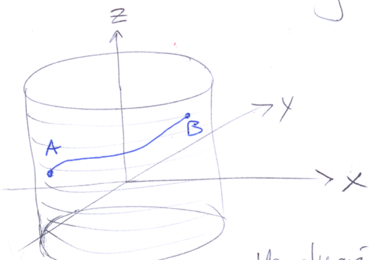
Hér höfum við vísvitandi
ekki tiltekið viðmiðunar-
punkt mættis.

$$\bar g(z) = -\hat e_z\frac{d\Phi}{dz} = +\frac{2GM}{R^2}\hat e_z\left\{\frac{z}{\sqrt{R^2+z^2}} - 1\right\}$$

$$= -\frac{2GM}{R^2}\hat e_z\left\{\frac{\sqrt{R^2+z^2} - z}{\sqrt{R^2+z^2}}\right\}$$

Ef $R \to \infty$ þarf að minka að $\frac{M}{R^2}$ er fasti
og í ljós kemur að $\bar{g}$ verður óháð z
Enginn lengdarstaði er eftir í kerfinu til að miða z við, og krafturinn er allsstaðar í sömu átt

06-04 Stysta leið milli tveggja punkta á sívalningi er um gorm feril



Geisli fastur $\rho = a$

Í kartísum hnitum er leiðarfrymið

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Hentugri eru sívalningshnit

$$x = a\cos\phi \qquad z = z$$
$$y = a\sin\phi$$

$$dx = -a\sin\phi \cdot d\phi \qquad dz = dz$$
$$dy = +a\cos\phi \cdot d\phi$$

$$\to ds = \sqrt{(a\,d\phi)^2 + (dz)^2} = \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2} \cdot d\phi$$

Þurfum að lágmarka

$$S = \int_A^B d\phi \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2} = \int_A^B d\phi\, f\left(z, \frac{dz}{d\phi}\right)$$

en $f = f\left(\frac{dz}{d\phi}\right)$ og z er fest í a

Athugið að heildarafleiðan $\frac{dz}{d\phi}$ lýsir því hvernig z breytist á leiðinni þegar ϕ - breytist. $\frac{\partial z}{\partial \phi} = 0$, en $\frac{dz}{d\phi} \neq 0$

Höfum engar upplýsingar um $\frac{dz}{d\phi}$, þar sem við höfum enga stikun fyrir leiðina

$\to$ notum að S lágmarkast þegar

$$\underbrace{\frac{\partial f}{\partial z}}_{=0} - \frac{\partial}{\partial \phi}\frac{\partial f}{\partial\left(\frac{dz}{d\phi}\right)} = 0 \quad \text{jafna Eulers}$$

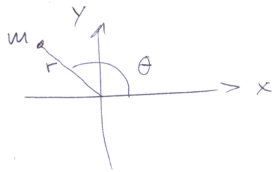
$$\to \frac{\partial}{\partial \phi}\frac{\left(\frac{dz}{d\phi}\right)}{\sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}} = 0 \quad \to \quad \frac{\left(\frac{dz}{d\phi}\right)}{\sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}} = \text{fasti} = C$$

$$\to \underbrace{\left(\frac{dz}{d\phi}\right)}_{} = \sqrt{\frac{C^2}{1-C^2}}\, a = \underbrace{\text{fasti}}_{}$$

jafnan fyrir einsleitum gormi

7.4 Ögn í sléttu með krafti $f = -Ar^{\alpha-1}$ að miðju
$A, \alpha > 0$

Vel pólhnit sem alhnit, $\bar{f} = -A r^{\alpha-1} \cdot \hat{e}_r$



$\bar{f} = -\bar{\nabla} U \to U = \frac{A}{\alpha} r^\alpha$

þar sem við veljum að $U(0) = 0$

$$T = \frac{m}{2}\left\{\dot{x}^2 + \dot{y}^2\right\}$$
$$= \frac{m}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2\right\}$$

$$L = T - U$$
$$= \frac{m}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2\right\} - \frac{Ar^\alpha}{\alpha}$$

Notum Euler-Lagrange jöfnur til að finna hreyfijöfnur

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \qquad \boxed{q_1 = r \quad q_2 = \theta}$$

2

r:

$$mr\dot{\theta}^2 - Ar^{\alpha-1} - m\ddot{r} = 0$$

eða

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 + Ar^{\alpha-1} = 0}$$

θ:

$$\boxed{-\frac{d}{dt}\left\{mr^2\dot{\theta}\right\} = 0}$$

Hverfiþunginn

$$\bar{L} = \bar{r}\times\bar{p} = r\hat{e}_r \times \left\{\bar{p}_r + \bar{p}_\theta\right\}$$

$$= r\hat{e}_r \times \bar{p}_\theta$$

$$\to |\bar{L}| = l = r\cdot mr\dot{\theta} = mr^2\dot{\theta}$$

Því er hreyfijafnan fyrir θ staðfesting á að hverfiþunginn sé ~~varð~~ varðveittur.

Hverfiþunginn er fasti

$$\to mr^2\dot{\theta} = l \text{ : fasti}$$

notum í hreyfij. fyrir r

$$\boxed{m\ddot{r} - \frac{l^2}{mr^3} + Ar^{\alpha-1} = 0}$$

↑ losnum við breytuna $\dot{\theta}$

Margföldum með $\dot{r}$

$$m\ddot{r}\dot{r} - \frac{\dot{r}l^2}{mr^3} + A\dot{r}r^{\alpha-1} = 0$$

3

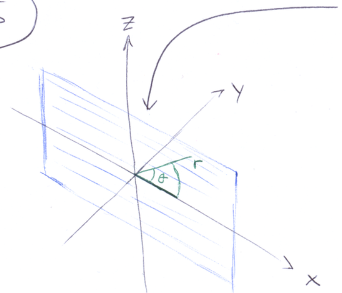
Sjáum að jafnan er heildarafleiða

$$\frac{d}{dt}\left\{\frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} + \frac{A}{\alpha}r^\alpha\right\} = 0$$

$$T + U = E$$

↑ heildarorkan, sem er þá varðveitt

7-5



Ögn hreyfist í lóðréttri sléttu í þyngdarsviði, með kraftinn $f = -Ar^{\alpha-1}$ að miðju í lóðréttu sléttunni

Í sléttunni notum við pólhnit

þ.a. $T = \frac{m}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2\right\}$

4

$$U = \frac{A}{\alpha}r^\alpha + mg\,r\sin\theta$$

sjá skýringarmynd á síðu 3

$$\to \boxed{L = \frac{m}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2\right\} - \frac{A}{\alpha}r^\alpha - mgr\sin\theta}$$

finnum hreyfijöfnur með Euler-Lagrange $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$

$\dot{r}$:

$$mr\dot{\theta}^2 - Ar^{\alpha-1} - mg\sin\phi - m\ddot{r} = 0$$

eða

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 + Ar^{\alpha-1} + mg\sin\theta = 0}$$

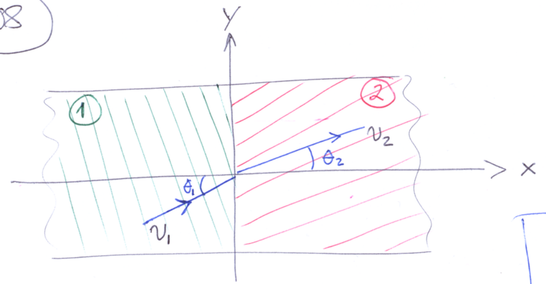
θ:

$$\boxed{mgr\cos\theta - \frac{d}{dt}\left\{mr^2\dot{\theta}\right\} = 0}$$

Hverfiþunginn í lóðréttu sléttunni er ekki varðveittur vegna vogis þyngdarkraftsins

5

7-08



$$U = \begin{cases} U_1 & \text{ef } x<0 \\ U_2 & \text{ef } x>0 \end{cases}$$

$$\boxed{L = \frac{m}{2}\left\{\dot{x}^2 + \dot{y}^2\right\} - U(x)}$$

Notum að $U(x) = \Theta(x)U_2 + \Theta(-x)U_1$

$$\Theta(x) = \begin{cases} 1 & \text{ef } x>0 \\ 0 & \text{ef } x<0 \end{cases}$$

↑ þrepafall Heavisides

Ennfremur $\Theta'(x) = \delta(x)$

↑ δ-fall Diracs

$$\int_a^b dx\,\delta(x) = 1 \quad \text{ef} \quad a<0<b$$

$$\int_a^b dx\, f(x)\delta(x) = f(0)$$

(6)

Euler-Lagrange gefur

$$\boxed{\begin{aligned} m\ddot{x} + \frac{dU}{dx} &= 0 \quad (1) \\ m\ddot{y} &= 0 \quad (2) \end{aligned}}$$

Notum að $\ddot{x} = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = \frac{dv_x}{dx} v_x$

Jafna (2) gefur að v_y sé alltaf fasti $\left(m\frac{dv_y}{dt} = 0\right)$

líka yfir stilin í $x=0$

skoðum jöfnu (1) yfir stilin, hún jafngildir

$$m\ddot{x} = -\delta(x)(U_2 - U_1) \qquad \text{þ.s. } \frac{dU}{dx} = \delta(x)(U_2-U_1)$$

$$\rightarrow \quad v_x \frac{dv_x}{dx} = -\delta(x)\frac{(U_2-U_1)}{m}$$

(7)

Heildum

$$\int_{v_{x1}}^{v_{x2}} v_x dv_x = -\frac{(U_2-U_1)}{m}\int_{x_1}^{x_2} dx\, \delta(x)$$

$$\frac{1}{2}\left\{v_{x2}^2 - v_{x1}^2\right\} = -\frac{(U_2-U_1)}{m}$$

eða

$$\frac{m}{2}v_{x1}^2 + U_1 = \frac{m}{2}v_{x2}^2 + U_2 \qquad (3)$$

Jafna (2) leiðir til $m\dot{y}_1 = m\dot{y}_2$, sem má umrita sem

$$\frac{1}{2}mv_{y1}^2 = \frac{1}{2}mv_{y2}^2 \qquad (4)$$

(3) og (4) má leggja saman

$$\rightarrow \quad \frac{1}{2}mv_1^2 + U_1 = \frac{1}{2}mv_2^2 + U_2 \qquad (5)$$

(8)

Munum síðan að $mv_{y1} = mv_{y2}$ má umrita sem

$$v_1 \sin\theta_1 = v_2 \sin\theta_2 \qquad (6)$$

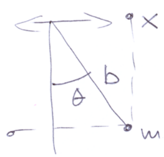
Setjum (6) og (5) saman

$$\rightarrow \quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \sqrt{\frac{v_1^2 + (U_1-U_2)\frac{2}{m}}{v_1^2}}$$

$$= \sqrt{1 + \frac{U_1-U_2}{\frac{m}{2}v_1^2}} = \sqrt{1 + \frac{U_1 - U_2}{T_1}}$$

(9)

7-16 Einfaldur pendúll hengdur upp í $x = a\sin(\omega t)$



$x = a\sin(\omega t)$, veljum hnitin fyrir m

$$x = a\sin(\omega t) + b\sin\theta$$

$$y = -b\cos\theta$$

$$\rightarrow \quad \dot{x} = a\omega\cos(\omega t) + b\dot{\theta}\cos\theta$$

$$\dot{y} = b\dot{\theta}\sin\theta$$

$$L = T - U = \frac{m}{2}\left\{\dot{x}^2 + \dot{y}^2\right\} + mgy$$

því fæst

$$L = \frac{m}{2}\left\{(a\omega)^2\cos^2(\omega t) + 2ab\omega\dot{\theta}\cos(\omega t)\cos\theta + (b\dot{\theta})^2\right\} + mgb\cos\theta$$

Eina raunverulega breytan sem eftir er, er θ

Notum Euler-Lagrange

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = 0$$

$$-\frac{m}{2}2ab\omega\dot\theta\cos(\omega t)\sin\theta - mgb\sin\theta$$
$$-\frac{d}{dt}\left\{\frac{m}{2}2ab\omega\cos(\omega t)\cos\theta + \frac{m}{2}2b^2\dot\theta\right\} = 0$$

$$-ab\omega^2\sin(\omega t)\cos\theta - ab\omega\dot\theta\cos(\omega t)\sin\theta + b^2\ddot\theta$$
$$= ab\omega\dot\theta\cos(\omega t)\sin\theta - gb\sin\theta$$

$$\rightarrow \quad \boxed{\ddot\theta + \frac{g}{b}\sin\theta - \frac{a\omega^2}{b}\sin(\omega t)\cos\theta = 0}$$

Gerum víddarlaust

$$\ddot\theta = \frac{d^2\theta}{dt^2} = \omega^2\frac{d^2\theta}{d(t\omega)^2} = \ddot\theta\cdot\omega^2$$, og notum $t\omega$ sem víddarlausa tímabreytu

því verður hreyfijafnan

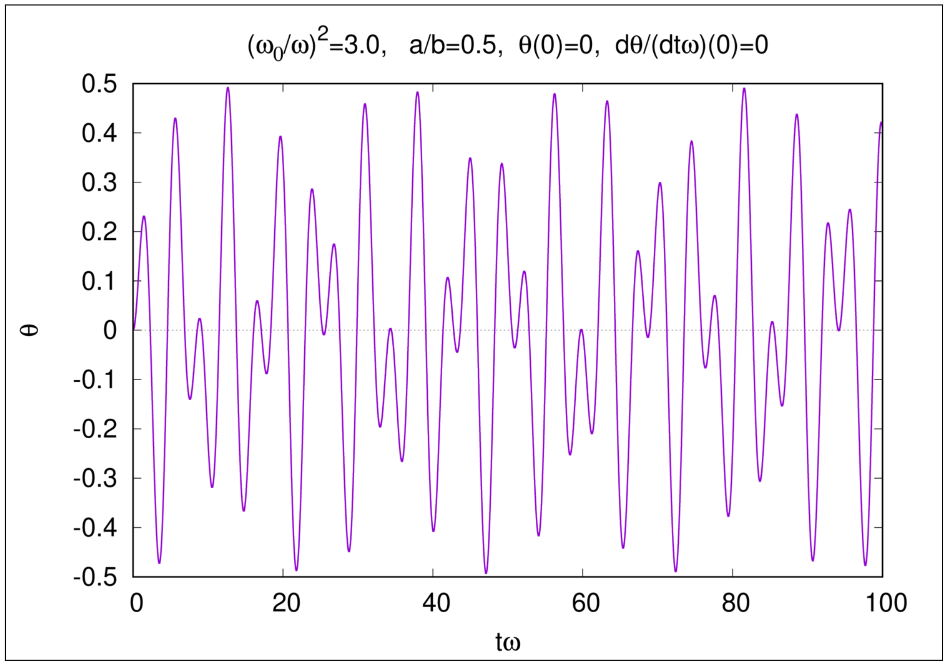
$$\ddot\theta + \left(\frac{\omega_0}{\omega}\right)^2\sin\theta - \frac{a}{b}\sin(t\omega)\cos\theta = 0$$

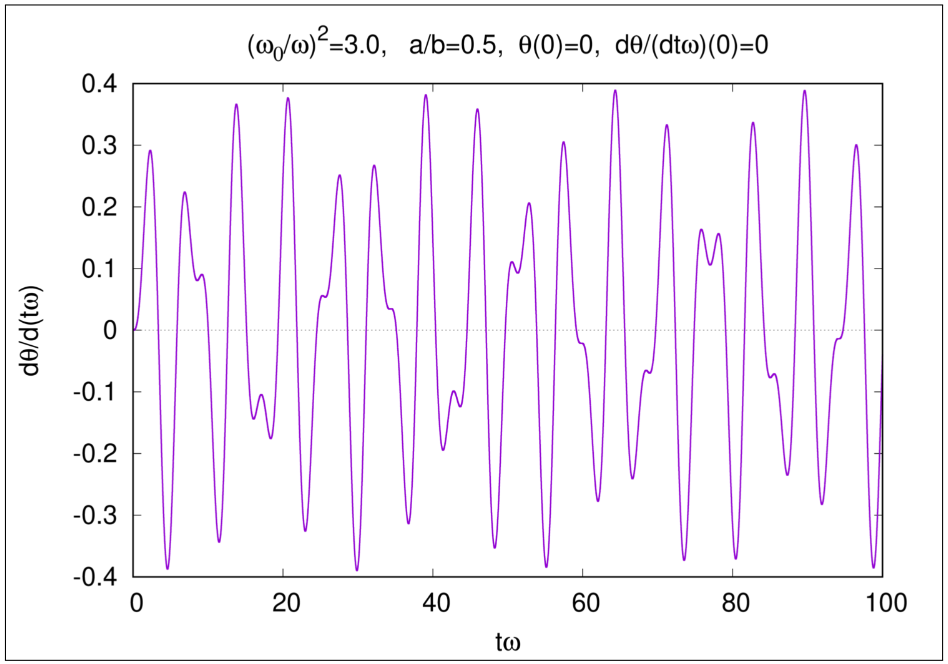
þar sem $\omega_0^2 = \frac{g}{b}$

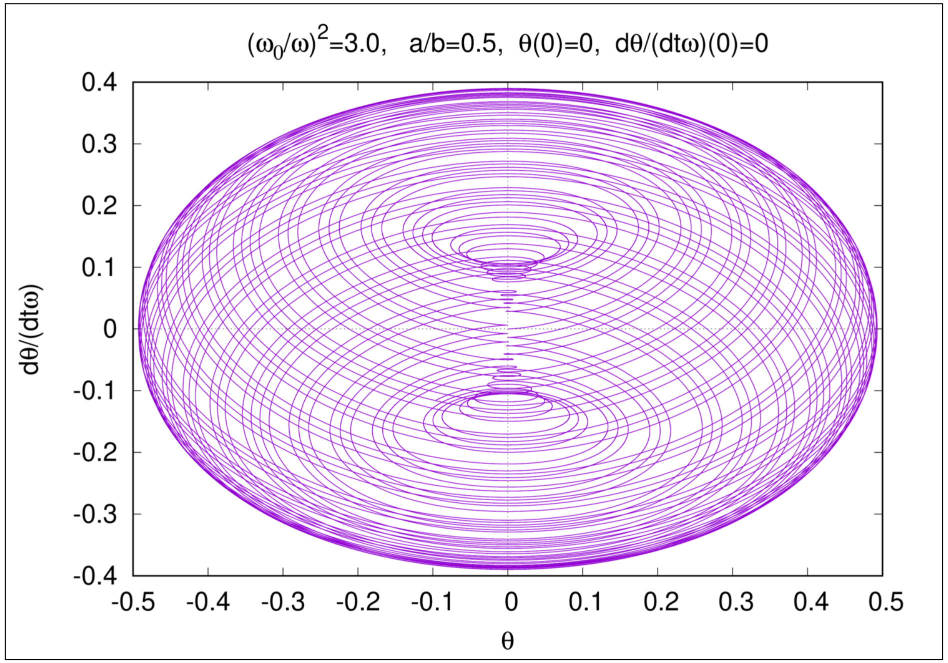
þá eru einu stikarnir eftir $\frac{\omega_0}{\omega}$: hlutfall náttúrulegu horntíðni pendúlsins og horntíðni ytri þvingunarinnar

og $\frac{a}{b}$: hlutfall lengdanna

Sjá myndir á næstu þremur síðum

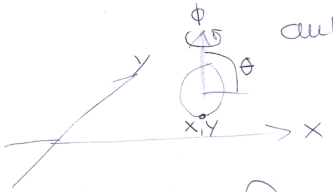






07-01 Skífa veltur á láréttum fleti

Alhnit? Við þurfum t.d. x og y - stöðuhnit auk hornanna θ og ϕ til að lýsa stöðu hennar algerlega.



Veltuskilyrði : Veltunni og „spunanum" z-ásinn má lýsa með skorðunum

$$R\,d\theta = dx\cdot\cos\phi + dy\cdot\sin\phi$$

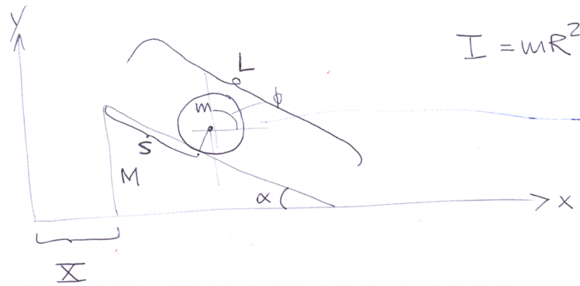
í stefnu ϕ þ.a. $\frac{dy}{dx} = \tan\phi$

Þessi skilyrði er ekki hægt að heilda t.o. finna $f(\theta,\phi,x,y)=0$ (mynda ekki heildarafleiðu) → nonholonomic

x,y ákvarða ekki θ og ϕ

7-6 Gjörð með m og R veltur niður skáplan með M og α m.v. núningslausa sléttu

$$I = mR^2$$



Sjáum að best er að velja upphaf ϕ þ.a. $s = R\phi$

Fyrir massamiðju gjarðar

$$x = X + s\cos\alpha + R\sin\alpha$$
$$y = R\cos\alpha + (L-s)\sin\alpha$$

Hreyfiorka gjarðar er þá

$$T_g = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right) + \frac{I}{2}\dot{\phi}^2$$
$$= \frac{m}{2}\left\{\left(\dot{X} + \dot{s}\cos\alpha\right)^2 + \left(-\dot{s}\sin\alpha\right)^2\right\} + \frac{m}{2}\left(R\dot{\phi}\right)^2$$

Notum $s = R\phi$ til að losa okkur við $\dot{\phi}$

$$\rightarrow T_g = \frac{m}{2}\left\{2\dot{s}^2 + \dot{X}^2 + 2\dot{X}\dot{s}\cos\alpha\right\}$$

Alhnitin eru í raun X og s

Fyrir skáplanið gildir

$$T_s = \frac{M}{2}\dot{X}^2$$

þá er heildar hreyfiorkan

$$T = m\dot{s}^2 + \frac{m+M}{2}\dot{x}^2 + m\dot{x}\dot{s}\cos\alpha$$

Mættisorkan er

$$U = mgy = mg\{R\cos\alpha + (L-s)\sin\alpha\}$$

og við sjáum að

$$L = T - U = m\dot{s}^2 + \frac{m+M}{2}\dot{x}^2 + m\dot{x}\dot{s}\cos\alpha$$
$$- mg\{R\cos\alpha + (L-s)\sin\alpha\}$$

Tvö atriði, notum Euler-Lagrange

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

til að finna

s:

$$mg\sin\alpha - \frac{d}{dt}\left\{2m\dot{s} + m\dot{x}\cos\alpha\right\} = 0$$

$$\rightarrow \boxed{2m\ddot{s} + m\ddot{x}\cos\alpha - mg\sin\alpha = 0} \quad (1)$$

x:

$$-\frac{d}{dt}\left\{2\frac{m+M}{2}\dot{x} + m\dot{s}\cos\alpha\right\} = 0$$

$$\rightarrow \boxed{(m+M)\ddot{x} + m\ddot{s}\cos\alpha = 0} \quad (2)$$

p.s. jafna (2) er óháð, má einangra

$$\left\{2 - \frac{m\cos^2\alpha}{m+M}\right\}\ddot{s} = g\sin\alpha$$

$$\ddot{x} = -\frac{mg\sin\alpha\cos\alpha}{2(m+M) - m\cos^2\alpha}$$

þegar $M \to \infty$ stöðast skáplanið, en gjörðin veltir ein faldboga

og þar

$$\frac{d}{dt}\{\underbrace{(m+M)\dot{x}}_{} + \underbrace{m\dot{s}\cos\alpha}_{}\} = 0$$

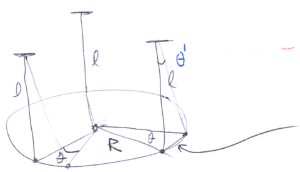
heildar x-þáttur skriðþunga kerfis

x-þáttur skriðþunga gjarðar miðað við skáplan

vegna þess að engir kraftar verka í x-átt

07-20

Hringur hengdur upp í þrem punktum, jafnhliða



snúið um lítið horn um miðju sína (vil ekki finna θ'), en þurfum strenglengd

lengd strengs er $2R\sin\left(\frac{\theta}{2}\right)$, en gjörðin hækkar þegar henni er snúið

Beinlengd niður á hana verður

$$l \rightarrow l^2 - \left(2R\sin\left(\frac{\theta}{2}\right)\right)^2$$

Mælum hæð gjarðar í upphafi $z = -l$, notum $z(\theta)$ og setjum $z(0) = -l$

$$\{z(\theta)\}^2 = l^2 - \left(2R\sin\left(\frac{\theta}{2}\right)\right)^2$$

$$\rightarrow z(\theta) = -\sqrt{l^2 - (2R)^2\sin^2\left(\frac{\theta}{2}\right)}$$

②

Fyrir lítið horn fæst þá $\sin^2(\frac{\theta}{2}) \simeq \frac{\theta^2}{4} + \cdots$

$$\to z(\theta) \simeq -\ell\sqrt{1 - \frac{R^2\theta^2}{\ell^2}} = -\ell\sqrt{1-\left(\frac{R\theta}{\ell}\right)^2}$$

$$\simeq -\ell\left\{1-\frac{1}{2}\left(\frac{R\theta}{\ell}\right)^2\right\}$$

því notum við $U = mgz \simeq -mg\ell\left\{1-\frac{1}{2}\left(\frac{R\theta}{\ell}\right)^2\right\}$

Massamiðja gjarðarinnar er kyrr, nema hvað hún getur fært til í z-átt

$$T = \frac{I}{2}\dot{\theta}^2 + \frac{m}{2}\dot{z}^2, \quad I = mR^2$$

③

$$\dot{z} = \frac{dz}{dt} = \frac{dz}{d\theta}\frac{d\theta}{dt} = \frac{R^2\theta}{\ell}\dot{\theta}$$

$$\to T = \frac{m}{2}(R\dot{\theta})^2 + \frac{m}{2}\frac{(R\theta)^2\dot{\theta}^2}{\ell^2}$$

$$\to L = \frac{m}{2}(R\dot{\theta})^2 + \frac{m(R\theta)^2\dot{\theta}^2}{2\ell^2} + mg\ell\left\{1-\frac{1}{2}\left(\frac{R\theta}{\ell}\right)^2\right\}$$

Þetta fall Lagrange verður óekins fyrir hreintóna sveifil ef þessi liður ~ 0. Hlutfallið milli hans og fyrsta liðsins er $\frac{(R\theta)^2}{\ell^2}$ sem við leyfum okkur að jafna við núll þegar það margfaldast við $\frac{m}{2}(R\dot{\theta})^2$

$$L \simeq \frac{m}{2}(R\dot{\theta})^2 - \frac{m}{2}\frac{g}{\ell}(R\theta)^2 + \text{fasti}$$

④

Berum saman við

$$L = \frac{m}{2}(R\dot{\theta})^2 - \frac{m}{2}\omega_0^2(R\theta)^2$$

fyrir hreintóna sveiflu, þar af leiðir að hreyfijafnan verður

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

og $\omega_0 = \sqrt{\frac{g}{\ell}}$

⑤

07-32

$U(r) = -\frac{k}{r}$, Hreyfing í mættinu í kúluhnitum

Notum jöfnu (1.100) á bls. 32 í bók

$$\bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\dot{\phi}\hat{e}_\phi$$

til að umskrifa

$$L = T - U = \frac{m}{2}v^2 + \frac{k}{r}$$

sem

$$L = \frac{m}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\cdot\dot{\phi})^2\right\} + \frac{k}{r}$$

þar sem við notum alhnitin r, θ og ϕ. Samsvarandi skriðþungar eru því

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\cdot\sin\theta$$

Fall Hamiltons er þá

$$H = p_r\dot r + p_\theta\dot\theta + p_\phi\dot\phi - L$$

$$= \frac{m}{2}\left\{\dot r^2 + (r\dot\theta)^2 + (r\sin\theta\cdot\dot\phi)^2\right\} - \frac{k}{r}$$

$$= \left\{\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2m(r\sin\theta)^2}\right\} - \frac{k}{r}$$

Hreyfijöfnur Hamiltons

$$\dot q_k = \frac{\partial H}{\partial p_k}$$

$$-\dot p_k = \frac{\partial H}{\partial q_k}$$

$$\dot r = \frac{p_r}{m}$$

$$\dot\theta = \frac{p_\theta}{mr^2}$$

$$\dot\phi = \frac{p_\phi}{m(r\sin\theta)^2}$$

eins og raunin var komið á síðu 5

$$\dot p_r = -\frac{k}{r^2} + \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3\sin^2\theta}$$

$$\dot p_\theta = \frac{p_\phi^2}{mr^2(\sin^2\theta}\cdot\frac{\cos\theta}{\sin\theta)} = \frac{p_\phi^2\cot\theta}{m(r\sin\theta)^2}$$

$$\dot p_\phi = 0$$

← p_ϕ er fasti, H er ekki fall af ϕ ← 4 breytur

því er í raun $H = H(r,\theta,p_r,p_\theta)$

6 fyrsta stigs hreyfijöfnur fyrir r, θ, ϕ ásamt p_r, p_θ, p_ϕ

Fjórvítt fasarúm

$$H = \left\{\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{\text{fasti}}{(r\sin\theta)^2} - \frac{k}{r}\right\}$$

Skoðum ofanvarp á r-p_r-sléttu fyrir breytilega θ

Orkuvarðveisla $H = E$

$$\to p_r = \sqrt{2mE - \frac{p_\theta^2\sin^2\theta + \text{fasti}}{(r\sin\theta)^2} + \frac{2mk}{r}}$$

07-24

Pendúll sem styttist $\frac{dl}{dt} = -\alpha$ = fasti

Veljum alhnit θ, en munum að staðsetningarhnit massa er $l\theta$

$$T = \frac{m}{2}\left(\frac{d}{dt}(l\theta)\right)^2 = \frac{m}{2}\left\{(l\dot\theta)^2 + (\dot l)^2\right\} = \frac{m}{2}\left\{(l\dot\theta)^2 + (\alpha)^2\right\}$$

$$U = -mgl\cos\theta$$

$$\to L = \frac{m}{2}\left\{(l\dot\theta)^2 + (\alpha)^2\right\} + mgl\cos\theta$$

$\bar v = \dot{\bar r} = \dot l\hat e_r + l\dot\theta\hat e_\theta$ og $\hat e_r\cdot\hat e_\theta = 0$

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = ml^2\dot\theta$$

þá er hægt að mynda $H = p_\theta\dot\theta - L$

$$H = ml^2\dot\theta^2 - \frac{m}{2}\left\{(l\dot\theta)^2 + (\alpha)^2\right\} - mgl\cos\theta$$

$$= \frac{p_\theta^2}{2ml^2} - \frac{m}{2}(\alpha)^2 - mgl\cos\theta$$

Heildarorkan er

$$E = T + U = \frac{m}{2}\left\{(l\dot\theta)^2 + (\alpha)^2\right\} - mgl\cos\theta$$

$$= \frac{p_\theta^2}{2ml^2} + \frac{m}{2}(\alpha)^2 - mgl\cos\theta$$

$$\to E \neq H$$

Skoðum beint hreyfijöfnuna, fundna með L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}\left\{ml^2\dot\theta\right\} + mgl\sin\theta = 0$$

$$ml^2\ddot\theta + 2\dot\theta ml\dot l + mgl\sin\theta = 0$$

$$ml^2\ddot\theta - 2\dot\theta ml\alpha + mgl\sin\theta = 0$$

$$\ddot\theta - \frac{2\alpha}{l}\dot\theta + \frac{g}{l}\sin\theta = 0$$

Ef við hefðum smáar sveiflur

$$\boxed{\ddot\theta - \underbrace{\frac{2\alpha}{l}\dot\theta} + \underbrace{\frac{g}{l}(1+0)\theta} = 0}$$

orkudeltinn í pendúlinn (stöðuorka hans eykst)

$$\omega_0 = \sqrt{\frac{g}{l}}$$

orkudælingin inn lætur hann sveiflast hraðar

Varúð, því jafnan lengir á sér, $l = l(t)$!

7-25 Gormferill $z = k\theta$, r = fasti $= b$

Ögn í þyngdarsviði hreyfist á gormferli

Best að nota sívalningshnit r, θ, z

Almennt gildi þá

$$T = \frac{m}{2}\left\{\dot r^2 + (r\dot\theta)^2 + \dot z^2\right\}, \quad U = mgz$$

→ Með þessum brautarskorðum

$r = b$, $\dot r = 0$, höldum z, en setjum $\theta = \frac{z}{k}$

$$\to \dot\theta = \frac{\dot z}{k} \quad \text{og} \quad (r\dot\theta)^2 = b^2\frac{\dot z^2}{k^2}$$

$$\to L = \frac{m}{2}\left\{\dot z^2 + \frac{b^2}{k^2}\dot z^2\right\} - mgz$$

$$= \frac{m}{2}\left\{1 + \frac{b^2}{k^2}\right\}\dot z^2 - mgz$$

Eina alhnitið er því z

$$p_z = \frac{\partial L}{\partial \dot z} = m\left(1 + \frac{b^2}{k^2}\right)\dot z$$

svo

$$\dot z = \frac{p_z}{m\left(1 + \frac{b^2}{k^2}\right)}$$

Fall Hamiltons verður því

$$H = p_z \dot z - L = \boxed{\frac{p_z^2}{2m\left(1 - \left(\frac{b}{k}\right)^2\right)} + mgz}$$

Lítur út eins og ögn í þyngdarkrafti á lóðréttri hreyfingu með virka massann $m\left(1 - \left(\frac{b}{k}\right)^2\right)$

$$\dot p_z = -\frac{\partial H}{\partial z} = -mg$$

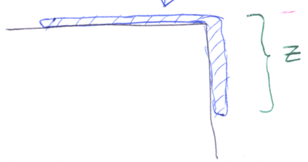
$$\dot z = \frac{\partial H}{\partial p_z} = \frac{p_z}{m\left(1 + \left(\frac{b}{k}\right)^2\right)}$$

$$\to \boxed{\left(1 + \left(\frac{b}{k}\right)^2\right)\ddot z = g}$$

Eins og giska mátti á þegar H eða L er skoðað

07-39

Mjög þjáll kapall, heildarlengd b



Setjum $U = 0$ fyrir hlutann á láréttu borðinu

$$U = g\int_z^0 dz'\,\rho(z')\,z'$$

þar sem $\rho(z) = \frac{m}{b}$ fasti

$$\to U = g\frac{m}{b}\frac{z'}{2}\Big|_z^0 = -g\frac{m}{b}\frac{z^2}{2}$$

Athugum síðan að allur kapallinn hreyfist samtímis → tregðumassinn er m, hreyfiorkan er óháð því hvort hreyfingin er lá- eða lóðrétt

$$\to T = \frac{m}{2}\dot z^2$$

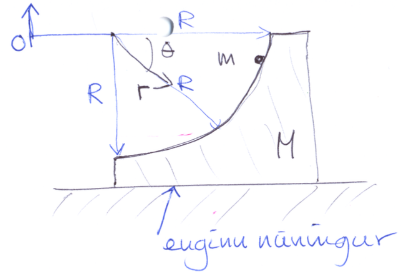
$$\rightarrow \quad L = \frac{m}{2}\dot{z}^2 + \frac{gm}{2b} z^2$$

Notum Euler Lagrange

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \quad \rightarrow \quad \frac{gmz}{b} - \frac{d}{dt}\left(m\dot{z}\right) = 0$$

$$\rightarrow \quad \boxed{m\ddot{z} - \frac{gmz}{b} = 0} \quad \text{eða} \quad \boxed{\ddot{z} - \frac{gz}{b} = 0}$$

07-34



a) Finna hreyfijöfnur m og M

b) Finna kraft M á m

Alhnitið fyrir M
$x_M = x$ (óþarft $y_M = 0$) $\rightarrow$ $x_m = r\cos\theta + x$, $y_m = -r\sin\theta$

Nægjanlegt til að nota almennt r í stað R þ.s. við viljum finna kraftinn á m í b-lið

$$T = T_M + T_m$$

$$T_M = \frac{M}{2}\dot{x}^2$$

$$T_m = \frac{m}{2}\left\{\dot{x}_m^2 + \dot{y}_m^2\right\} = \frac{m}{2}\left\{(\dot{r}\cos\theta - r\dot{\theta}\sin\theta + \dot{x})^2 + (-\dot{r}\sin\theta - r\dot{\theta}\cos\theta)^2\right\}$$

$$U = mgy_m = -mgr\sin\theta$$

$$T_m = \frac{m}{2}\left\{\dot{r}^2\cos^2\theta + \dot{x}^2 + r^2\dot{\theta}^2\sin^2\theta \cancel{-2r\dot{r}\dot{\theta}\sin\theta\cos\theta} + 2\dot{r}\dot{x}\cos\theta - 2r\dot{\theta}\dot{x}\sin\theta + \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta \cancel{+2\dot{r}r\dot{\theta}\cos\theta\sin\theta}\right\}$$

$$\rightarrow \quad T_m = \frac{m}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2 + 2\dot{r}\dot{x}\cos\theta - 2r\dot{\theta}\dot{x}\sin\theta\right\} + \frac{m}{2}\dot{x}^2$$

því verður

$$\boxed{L = \frac{(M+m)}{2}\dot{x}^2 + \frac{m}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2 + 2\dot{r}\dot{x}\cos\theta - 2r\dot{\theta}\dot{x}\sin\theta\right\} + mgr\sin\theta}$$

og jafnan fyrir brautarskorðum er

$$f(x,\theta,r) = r - R = 0$$

Alhnitin eru x, θ, og r

Hreyfijöfnurnar fyrir kerfið fást með Euler-Lagrange þegar við setjum $r = R$ og $\dot{r} = 0$, $\ddot{r} = 0$ (eftir standa x og θ)

$$L' = \left(\frac{M+m}{2}\right)\dot{x}^2 + \frac{m}{2}\left\{(R\dot{\theta})^2 - 2R\dot{\theta}\dot{x}\sin\theta\right\} + mgR\sin\theta$$

$$\frac{\partial L'}{\partial x} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}}\right) = 0 : \quad \frac{d}{dt}\left\{(M+m)\dot{x} - mR\dot{\theta}\sin\theta\right\} = 0$$

$$\rightarrow \quad \boxed{(M+m)\ddot{x} - mR\ddot{\theta}\sin\theta - mR\dot{\theta}^2\cos\theta = 0} \quad (1)$$

$$\frac{\partial L'}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\theta}}\right) = 0 : \quad -mR\dot{\theta}\dot{x}\cos\theta + mgR\cos\theta - \frac{d}{dt}\left\{mR^2\dot{\theta} - mR\dot{x}\sin\theta\right\} = 0$$

$$\rightarrow \quad mR^2\ddot{\theta} - mR\ddot{x}\sin\theta \cancel{- mR\dot{x}\dot{\theta}\cos\theta} \cancel{+ mR\dot{\theta}\dot{x}\cos\theta} - mgR\cos\theta = 0$$

$$\rightarrow \boxed{\ddot{\theta} - \frac{\ddot{x}}{R}\sin\theta - \frac{g}{R}\cos\theta = 0} \quad (2)$$

b) Brautarskorðurnar

$$f(r) = r - R = 0$$

skorðuliðir

Athugum $\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) + \lambda\frac{\partial f}{\partial r} = 0$

$$mr\dot{\theta}^2 - m\dot{\theta}\dot{x}\sin\theta + mg\sin\theta - \frac{d}{dt}\left\{m\dot{r} + m\dot{x}\cos\theta\right\} + \lambda = 0$$

$$\rightarrow mr\dot{\theta}^2 - \cancel{m\dot{\theta}\dot{x}\sin\theta} + mg\sin\theta - m\ddot{r} - m\ddot{x}\cos\theta + \cancel{m\dot{x}\dot{\theta}\sin\theta} + \lambda = 0$$

$$mr\dot{\theta}^2 + mg\sin\theta - m\ddot{r} - m\ddot{x}\cos\theta + \lambda = 0$$

$$\lambda = m\ddot{x}\cos\theta + m\ddot{r} - mg\sin\theta - mr\dot{\theta}^2$$

Setjum núna rétta braut, $r = R$, $\dot{r}, \ddot{r} = 0$

$$\rightarrow \lambda = m\ddot{x}\cos\theta - mg\sin\theta - mR\dot{\theta}^2 \quad (3)$$

Notum núna (1) og (2)

$$\ddot{x} = \mu R\ddot{\theta}\sin\theta + \mu R\dot{\theta}^2\cos\theta, \quad \mu = \frac{m}{M+m}$$

$$\ddot{\theta} = \frac{\ddot{x}}{R}\sin\theta + \frac{g}{R}\cos\theta$$

$$\ddot{x} = \mu R\left\{\frac{\ddot{x}}{R}\sin\theta + \frac{g}{R}\cos\theta\right\}\sin\theta + \mu R\dot{\theta}^2\cos\theta$$

$$= \mu\ddot{x}\sin^2\theta + g\mu\cos\theta\sin\theta + \mu R\dot{\theta}^2\cos\theta$$

$$\rightarrow \ddot{x}(1 - \mu\sin^2\theta) = \mu R\dot{\theta}^2\cos\theta + g\mu\cos\theta\sin\theta$$

$$\ddot{x} = \frac{\mu\cos\theta\left\{R\dot{\theta}^2 + g\sin\theta\right\}}{1 - \mu\sin^2\theta}$$

Notum í (3)

$$\lambda = \frac{m\mu\cos^2\theta\left\{R\dot{\theta}^2 + g\sin\theta\right\}}{1 - \mu\sin^2\theta} - mg\sin\theta - mR\dot{\theta}^2$$

$$= \frac{m\mu\cos^2\theta\left\{R\dot{\theta}^2 + g\sin\theta\right\} - mg\sin\theta + mg\mu\sin^3\theta - mR\dot{\theta}^2 + m\mu R\dot{\theta}^2\sin^2\theta}{1 - \mu\sin^2\theta}$$

$$= \frac{m\mu R\dot{\theta}^2 - mg\sin\theta - mR\dot{\theta}^2 + m\mu g\cos^2\theta\sin\theta + m\mu g\sin^3\theta}{1 - \mu\sin^2\theta}$$

$$\rightarrow \lambda = \frac{m\mu R\dot{\theta}^2 - mR\dot{\theta}^2 - mg\sin\theta + m\mu g\sin\theta}{1 - \mu\sin^2\theta}$$

$$= \frac{m(\mu - 1)\left\{g\sin\theta + R\dot{\theta}^2\right\}}{1 - \mu\sin^2\theta}$$

þarsem θ_0 er upphafshorn þar sem ögnin hefur engan hraða

$\dot{\theta}$ má finna frá orkuvarðveislu

$$\left(\frac{M+m}{2}\right)\dot{x}^2 + \frac{m}{2}\left\{R^2\dot{\theta}^2 - 2\dot{x}R\dot{\theta}\sin\theta\right\} - mgR\sin\theta = -mgR\sin\theta_0$$

En hér þarf að losna við $\dot{x}$, til þess þarf að heilda (1)

Skoðum (3)

$$\ddot{x} = \mu R\left\{\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right\} = \mu R\frac{d}{dt}\left\{\dot{\theta}\sin\theta\right\}$$

$$\frac{d}{dt}(\dot{x}) = \mu R\frac{d}{dt}\left\{\dot{\theta}\cdot\sin\theta\right\}$$

Því fæst $\dot{x} = \mu R \dot{\theta} \sin\theta$, og orkuvarðveislan

$$\left(\frac{M+m}{2}\right)\mu^2 R^2 \dot{\theta}^2 \sin^2\theta + \frac{m}{2}\left\{R^2\dot{\theta}^2 - 2\mu R^2 \dot{\theta}^2 \sin^2\theta\right\} - mgR\sin\theta = -mgR\sin\theta_0$$

$$\rightarrow \quad \dot{\theta}^2 = \frac{2g(\sin\theta - \sin\theta_0)}{R(1-\mu\sin^2\theta)}, \qquad \mu = \frac{m}{M+m}$$

Þetta þarf að nota í

$$\lambda = \frac{m(\mu-1)\{g\sin\theta + R\dot{\theta}^2\}}{1-\mu\sin^2\theta}$$

$$\lambda = \frac{m(\mu-1)}{(1-\mu\sin^2\theta)^2}\left\{g\sin\theta(1-\mu\sin^2\theta) + 2g(\sin\theta - \sin\theta_0)\right\}$$

$$= \frac{m(\mu-1)}{(1-\mu\sin^2\theta)^2}\left\{3g\sin\theta - g\mu\sin^3\theta - 2g\sin\theta_0\right\}$$

$$= \frac{m(\mu-1)g}{(1-\mu\sin^2\theta)^2}\left\{3\sin\theta - \mu\sin^3\theta - 2\sin\theta_0\right\}$$

Alkraftinn $Q_r = \lambda \frac{\partial f}{\partial r} = \lambda$ $\qquad \mu = \frac{m}{M+m}$

Ef $M \to \infty$, þá $M >>> m$ fæst $\lambda = -mg\{3\sin\theta - 2\sin\theta_0\}$

Ef síðan $\theta_0 = 0$ $\rightarrow$ $\lambda = -mg\,3\sin\theta$

8-32 Athuga stöðugleika hringbrauta fyrir kraftinn

$$F(r) = -\frac{k}{r^2}e^{-r/a}$$

Samkvæmt (8,93) verður að gilda að

$$\frac{F'(r)}{F(r)} + \frac{3}{r} > 0$$

$$\rightarrow \quad \frac{\left[\frac{2k}{r^3}e^{-r/a} + \frac{k}{ar^2}e^{-r/a}\right]}{-\frac{k}{r^2}e^{-r/a}} + \frac{3}{r} > 0$$

$$\rightarrow \quad \frac{2k + \frac{rk}{a}}{-kr} + \frac{3}{r} > 0 \qquad \Big| \qquad \frac{2 + \frac{r}{a}}{-r} + \frac{3}{r} > 0$$

$$\rightarrow \quad \frac{3 - 2 - \frac{r}{a}}{r} > 0 \quad \rightarrow \quad r < a$$

08-06 Tveir massar

$M = m_1 + m_2$

r_0 í upphafi, $v_1 = 0$, $v_2 = 0$

$r = |x_2 - x_1|$, $r < r_0$ þegar $t > 0$

Heildarorkan er þá stöðuorkan í upphafi

$$E_{total} = -G\frac{m_1 m_2}{r_0}$$

Síðar í tíma: $\qquad E_{total} = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 - G\frac{m_1 m_2}{r}$

Orkan er varðveitt, engir ytri kraftar -> heildar skriðþunginn er líka varðveittur (3)

$$\boxed{P_{total} = m_1\dot{x}_1 + m_2\dot{x}_2 = 0}$$ (1)

því agnirnar voru kyrrar í upphafi

Orkuvarðveislan:

$$\boxed{-G\frac{m_1m_2}{r_0} = \frac{m_1\dot{x}_1^2}{2} + \frac{m_2}{2}\dot{x}_2^2 - G\frac{m_1m_2}{r}}$$ (2)

Notum (1) $\to \quad m_1\dot{x}_1 + m_2\dot{x}_2 = 0 \to m_2\dot{x}_2 = -m_1\dot{x}_2$

í (2)

$$-Gm_1m_2\left\{\frac{1}{r_0} - \frac{1}{r}\right\} = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_1^2}{2m_2}\dot{x}_1^2$$

$$= \frac{m_1^2}{2}\left\{\frac{\dot{x}_1^2}{m_1} + \frac{\dot{x}_1^2}{m_2}\right\} = \frac{\dot{x}_1^2}{2}m_1^2\left\{\frac{m_2+m_1}{m_1m_2}\right\}$$

(4)

$$\dot{x}_1^2 = 2\frac{m_1m_2}{M}\frac{1}{m_1^2}\cdot Gm_1m_2\left\{\frac{1}{r} - \frac{1}{r_0}\right\}$$

$$= 2\frac{m_2^2}{M}G\left\{\frac{1}{r} - \frac{1}{r_0}\right\}$$

$$\to \quad \dot{x}_1 = m_2\sqrt{\frac{2G}{M}\left\{\frac{1}{r} - \frac{1}{r_0}\right\}}$$

og vegna (1) fæst því strax

$$\dot{x}_2 = -m_1\sqrt{\frac{2G}{M}\left\{\frac{1}{r} - \frac{1}{r_0}\right\}}$$

(08-20) Sýna að fyrir eind á sporbaug gildi (5)

"Ögn í þyngdarmætti"

$$\left\langle \left(\frac{a}{r}\right)^4 \cos\theta \right\rangle = \frac{\epsilon}{(1-\epsilon^2)^{5/2}}$$

tímameðaltal

samkvæmt (8-41)

(1) $\quad \frac{\alpha}{r} = 1 + \epsilon\cos\theta \quad , \quad \alpha = \frac{l^2}{\mu k} \quad , \quad \mu = \frac{m_1m_2}{m_1+m_2}$

og (8-42) $\qquad l = \mu r^2\dot{\theta} = \text{fasti}$

(2) $\quad a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2|E|} \qquad F = -\frac{k}{r^2}$

Notum (1) og (2) til að skrifa (6)

$$\frac{\alpha}{r} = 1 + \epsilon\cos\theta \quad , \quad \alpha = a(1-\epsilon^2)$$

$$\to \quad \frac{a(1-\epsilon^2)}{r} = 1 + \epsilon\cos\theta \quad \to \quad \frac{a}{r} = \frac{1+\epsilon\cos\theta}{(1-\epsilon^2)}$$

$$\to \quad \left(\frac{a}{r}\right)^4\cos\theta = \left[\frac{1+\epsilon\cos\theta}{(1-\epsilon^2)}\right]^4\cos\theta$$

$$\to \quad \left\langle\left(\frac{a}{r}\right)^4\cos\theta\right\rangle = \frac{1}{\tau}\int_0^\tau dt\left\{\frac{1+\epsilon\cos\theta}{(1-\epsilon^2)}\right\}^4\cos\theta$$

Eigum eftir að tengja θ og t

Annað lögmál Keplers (8-12) (7)

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \text{fasti}$$

og

$$\frac{dA}{dt} = \frac{\pi ab}{\tau} = \frac{\text{flötur}}{\text{tímalotu}}$$

$$\alpha = a(1-\epsilon^2)$$

$$dt = \frac{\tau}{\pi ab}\,dA = \frac{\tau}{\pi ab}\frac{r^2}{2}\,d\theta = \frac{\tau}{\pi ab}\frac{\alpha^2}{2(1+\epsilon\cos\theta)^2}\,d\theta$$

$$\to \left\langle \left(\frac{a}{r}\right)^4 \cos\theta \right\rangle = \frac{1}{\tau}\frac{1}{(1-\epsilon^2)^2}\frac{\tau}{\pi ab}\frac{a^2}{2}\int_0^{2\pi} d\theta\,\cos\theta(1+\epsilon\cos\theta)^2$$

(8)

$$= \frac{1}{(1-\epsilon^2)^2}\frac{a^2}{2\pi ab}\underbrace{\int_0^{2\pi} d\theta\,\cos\theta(1+\epsilon\cos\theta)^2}_{2\pi\epsilon}$$

$$= \frac{1}{(1-\epsilon^2)^2}\frac{a\epsilon}{b} = \frac{\epsilon}{(1-\epsilon^2)^{5/2}}$$

$$b = \frac{\alpha}{\sqrt{1-\epsilon^2}} = \frac{a(1-\epsilon^2)}{\sqrt{1-\epsilon^2}} = a(1-\epsilon^2)^{1/2}$$

(9)

08-10 Ef braut jarðar væri hringur, hvað gerðist ef massi sólar helmingaðist allt í einu

Hringur $\to T = \frac{m}{2}(\omega R)^2$

R: geisli brautar jarðar
m: massi jarðar
M: massi sólar

$$U = -G\frac{Mm}{R}$$

Fyrir hringbraut þarf þyngdarkrafturinn að vera jafn miðsóknarkraftinum, sem er nauðsynlegur fyrir hringbraut

$$m\omega^2 R = G\frac{Mm}{R^2}$$

$$\to \omega^2 = \frac{GM}{R^3}$$

(10)

$$\to T = \frac{m}{2}\frac{GMR^2}{R^3} = G\frac{Mm}{2R} = -\frac{1}{2}U$$

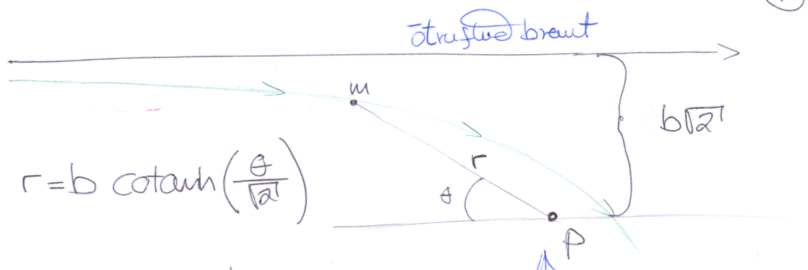
$$\to E = T + U = \frac{U}{2}$$

Ef massi sólar helmingast verður T óbreytt en U helmingast. Nýja orkan væri þá

$$E_2 = T + U_2 = T + \frac{U}{2} = 0$$

→ brautin breytist í fleygboga og jörðin yfirgæfi sólina

08-15



Sýna að $r = b\,\mathrm{cotanh}\left(\frac{\theta}{\sqrt{2}}\right)$

Hverfiþunginn um P er fasti

Í upphafi er hann

$$\ell = mv\,(b\sqrt{2})$$

hraði massans í mikilli fjarlægð, upphafshraði

Ef P er kraftmiðja með $F = -\frac{k}{r^5}$

og hverfiþunginn um P er $\frac{\sqrt{km}}{b}$

Til að víddin sé í lagi

$$\ell = \frac{\sqrt{km}}{b} = mv\,(b\sqrt{2})$$

hverfiþunginn er varðveittur

$\rightarrow v = \sqrt{\frac{k}{2m}}\,\frac{1}{b^2}$, og heildarorkan sem líka er varðveitt er upphafleg hreyfiorkan

$$E = \frac{m}{2}v^2 = \frac{m}{2}\,\frac{k}{2}\,\frac{1}{b^4 m} = \frac{k}{4b^4}$$

Gefið $F(r) = -\frac{k}{r^5}$

$$\rightarrow d\theta = \frac{\ell}{r^2}\,\frac{dr}{\sqrt{2m\left\{E + \frac{k}{4r^4} - \frac{\ell^2}{2mr^2}\right\}}}$$

$\frac{k}{4b^4}$ $\quad U(r)$ $\quad$ nota $\ell^2 = \frac{k}{b^2}$

$$d\theta = \frac{\sqrt{km}}{br^2}\,\frac{dr}{\sqrt{\frac{km}{2b^4} + \frac{km}{2r^4} - \frac{km}{b^2r^2}}} = b\sqrt{2}\,\frac{dr}{\sqrt{r^4 - 2b^2r^2 + b^4}}$$

$$= b\sqrt{2}\,\frac{dr}{\sqrt{(r^2-b^2)^2}} = -b\sqrt{2}\,\frac{dr}{r^2-b^2}$$

– greinin á rótinni er valin þ.s. θ vex þ. r minnkar samkvæmt myndinni á síðu (11) ......

Heildum

$$\theta = b\sqrt{2}\,\frac{1}{2b}\ln\left\{\frac{r+b}{r-b}\right\} + \theta_0 = \sqrt{2}\,\mathrm{ArCoth}\left(\frac{r}{b}\right) + \theta_0$$

þegar $r \to \infty$, sýnir myndin að $\theta_0 = 0$

Skoðum $r = b\coth\left\{\frac{\theta - \theta_0}{\sqrt{2}}\right\}$ $\quad$ $r \to \infty$ þegar $\theta \to 0$ ef $\theta_0 = 0$

$$\rightarrow r = b\coth\left(\frac{\theta}{\sqrt{2}}\right)$$

sjá mynd á síðu (15)

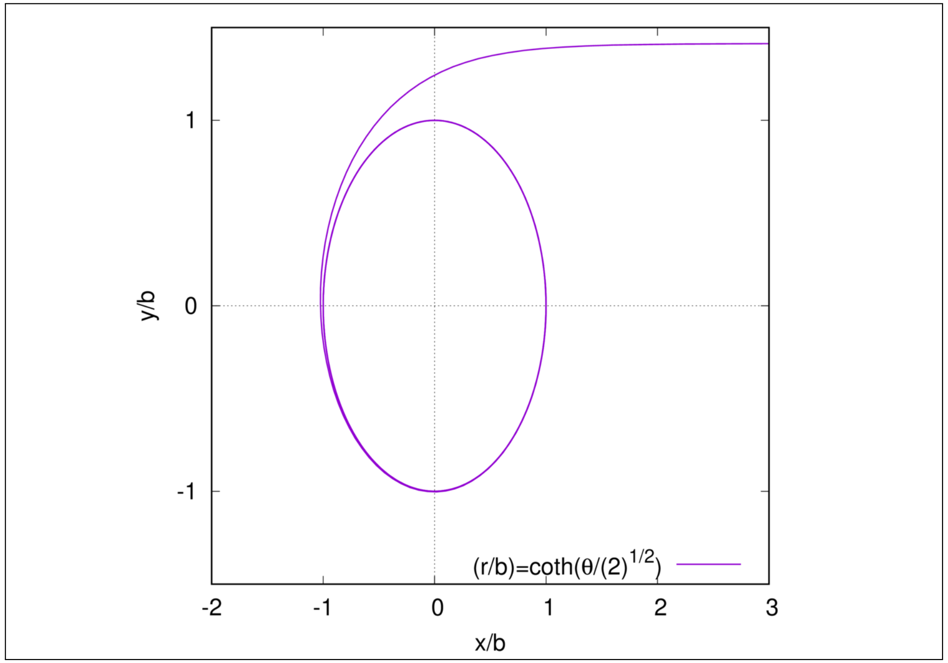
08-13 Skoða hreyfingu agnar í kraftsviði

$$F(r) = -\frac{k}{r^2} - \frac{\lambda}{r^3}, \quad k > 0,\ \lambda > 0$$

athuga tilfelli

$\lambda < \frac{\ell^2}{\mu}$ $\quad$ og $\lambda > \frac{\ell^2}{\mu}$

$\lambda = \frac{\ell^2}{\mu}$



16

Notum (8.20) þar sem $u = \frac{1}{r}$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}F\left(\frac{1}{u}\right)$$

Sem verður þá

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2 u^2}\left\{-ku^2 - \lambda u^3\right\}$$

$$\rightarrow \boxed{\frac{d^2u}{d\theta^2} + \left\{1-\frac{\mu\lambda}{\ell^2}\right\}u = \frac{\mu k}{\ell^2}}$$

sem er einföld línuleg afleiðujafna, en hægt er að ummynda hana í óhliðraða jöfnu

17

Sem sést með að skrifa

$$\frac{d^2u}{d\theta^2} + \left\{1-\frac{\mu\lambda}{\ell^2}\right\}\left\{u - \frac{(\mu k/\ell^2)}{\left(1-\frac{\mu\lambda}{\ell^2}\right)}\right\} = 0$$

og skilgreina nýtt fall $v = u - \underbrace{\frac{(\mu k/\ell^2)}{\left(1-\frac{\mu\lambda}{\ell^2}\right)}}_{\text{bara fasti}}$

þá fæst óhliðraða jafnan

$$\boxed{\frac{d^2v}{d\theta^2} + \left\{1-\frac{\mu\lambda}{\ell^2}\right\}v = 0} \quad (1)$$

Eins og jafnan fyrir hreintóna sveifilinn en $\beta^2 = \left\{1-\frac{\mu\lambda}{\ell^2}\right\}$ sem er sambærilegt við ω^2 í sveiflinum getur verið jákvætt, neikvætt, eða 0

18

$\underline{\frac{\mu\lambda}{\ell^2} < 1} \rightarrow \beta^2 > 0$

(1) hefur þá lausn

$v = A\cos(\beta\theta + \delta)$, veljum θ_0 þ.a. $\delta = 0$

$v = A\cos(\beta\theta) = u - \frac{\mu k}{\ell^2 - \mu\lambda}$, $u = \frac{1}{r}$

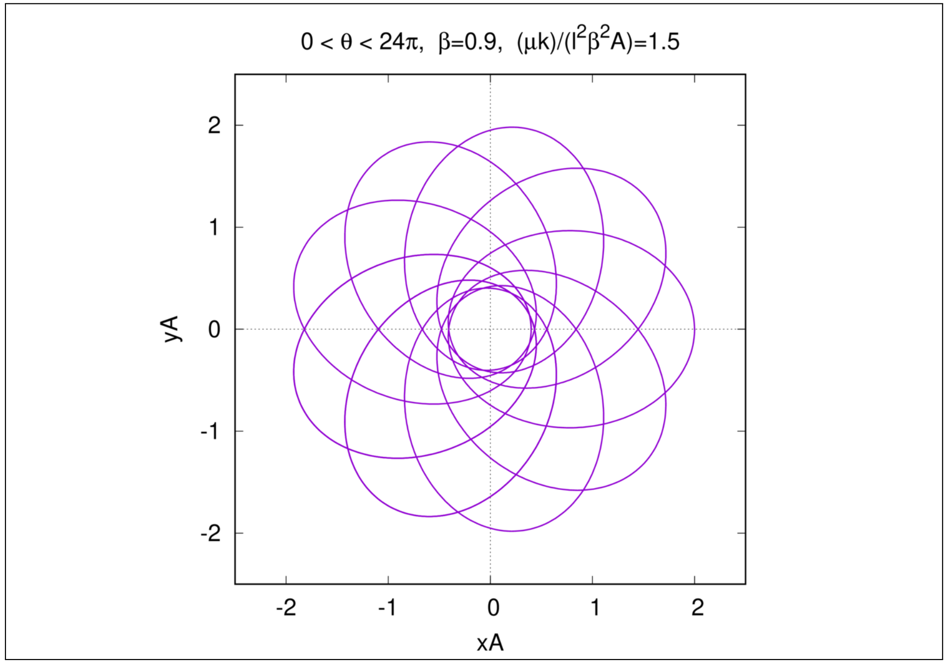
$$\rightarrow \boxed{\frac{1}{r} = A\cos(\beta\theta) + \frac{\mu k}{\ell^2 - \mu\lambda}}$$

$$\frac{1}{r} = A\cos(\beta\theta) + \frac{\mu k}{\ell^2\beta^2}$$

$$\frac{1}{rA} = \cos(\beta\theta) + \frac{\mu k}{l^2\beta^2 A}$$

$$\longrightarrow \quad rA = \frac{1}{\cos(\beta\theta) + \frac{\mu k}{l^2\beta^2 A}}$$

sjá snúning brautarinnar á síðu 20



$$\frac{\mu\lambda}{l^2} = 1 \longrightarrow \beta = 0$$

fyrir u verði jafnan þá

$$\frac{d^2u}{d\theta^2} = \frac{\mu k}{l^2} \longrightarrow u = \frac{\mu k}{2l^2}\theta^2 + A\theta + B = \frac{1}{r}$$

$\longrightarrow$ ögnin fellur inn í kraftmiðjuna með minnkandi gormhreyfingu ....

$$\frac{\mu\lambda}{l^2} > 1 \longrightarrow \beta^2 < 0$$

$$\boxed{u = A\cosh\left\{|\beta|\theta - \delta\right\}}$$

Lausnin verður

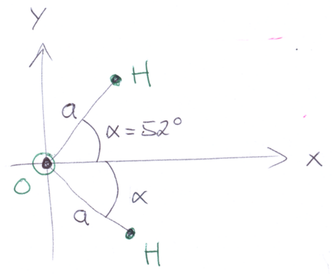
$$\frac{1}{r} = A\cosh\left\{|\beta|\theta\right\} + \frac{\mu k}{l^2 - \mu\lambda}$$

$\longrightarrow$ ögnin fellur í gormhreyfingu inn að kraftmiðju

09-07

H_2O, finna CM

1



spegilsamhverfa um x-ás

$\rightarrow Y_{CM} = 0$

$$X_{CM} = \frac{1}{M}\sum_{i=1}^{3} m_i x_i$$

$m_O = 16 m_H$

$$M = m_O + 2m_H = (16+2)m_H$$

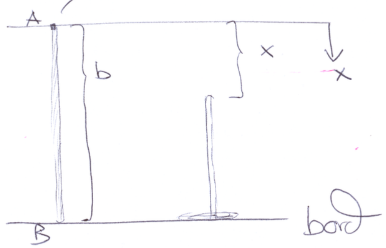
$$X_{CM} = \frac{1}{18 m_H}\cdot 2m_H \cdot a \cdot \cos\alpha$$

$$= \frac{a}{9}\cos\alpha \approx 0.068\cdot a$$

09-19

2

sleppt klukkan $t=0$, lengd b



↑ fyrir $t>0$

$\rho = \frac{M}{b}$

Hver er kraftur borðsins F á keðjuna sem fall af x?

Þegar hún hefur öll fallið er $\bar{F} = +\rho b g \hat{e}_x$ og massamiðjan kyrr $\rightarrow \frac{dp}{dt} = 0$, því kraftur borðsins er $-\bar{F}$

því er $\frac{dp}{dt} = \rho b g - F$ $\qquad p = \rho(b-x)\dot{x}$

3

$$\frac{dp}{dt} = \frac{d}{dt}\left\{\rho(b-x)\dot{x}\right\} = -\rho\dot{x}\dot{x} + \rho(b-x)\ddot{x}$$

Massamiðjan er í þyngdarsviði

$\rightarrow \boxed{\ddot{x} - g = 0}$

fellur úr kyrrstöðu með $x=0$

$\hookrightarrow E = 0$

Orkuvarðveisla $E = \frac{m}{2}\dot{x}^2 - mgx$,

$\rightarrow \boxed{\dot{x}^2 = 2gx}$

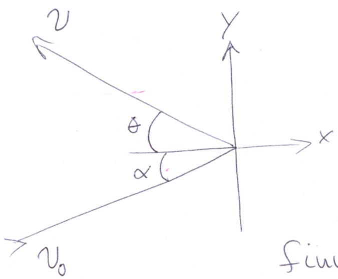
$$\frac{dp}{dt} = -\rho 2gx + \rho(b-x)g = \rho\left\{bg - 3gx\right\}$$

og áður

$\frac{dp}{dt} = \rho b g - F \Big\} \rightarrow \boxed{F = 3\rho g x}$, og $\bar{F} = -3\rho g x \hat{e}_x$

09-42

4



$\alpha = 30°$

$v_0 = 5$ m/s

$\epsilon = \frac{|v|}{|v_0|}$ $\qquad$ hér $\epsilon = 0{,}8$

finna v og θ

Engin breyting í hreyfingu í y-átt

$\rightarrow \underbrace{v_0 \sin\alpha}_{v_{y0}} = \underbrace{v\sin\theta}_{v_y}$ $\qquad \rightarrow v = v_0\frac{\sin\alpha}{\sin\theta}$

$v_y = \underbrace{v_0 \sin\alpha}_{\text{þekkt}}$

Vegna ófjaðrandi árekstrs, lýst með ϵ

$\rightarrow \left|\frac{v_x}{v_{x0}}\right| = \left|\frac{v\cos\theta}{v_0\cos\alpha}\right| = \epsilon$

(5)

$$v_x = \epsilon v_0 \cos\alpha$$

$$\to \quad v = \sqrt{v_x^2 + v_y^2} = v_0\sqrt{\epsilon^2\cos^2\alpha + \sin^2\alpha} = v_0 \cdot 0{,}8544$$

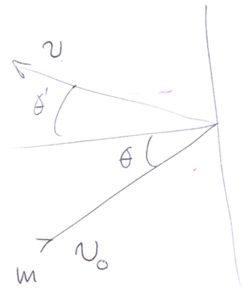
$$\frac{v_y}{v_x} = \frac{v_0 \sin\alpha}{\epsilon v_0 \cos\alpha} = \tan\theta$$

$$\to \quad \theta = \arctan\left\{\frac{\sin\alpha}{\epsilon\cos\alpha}\right\} = 0{,}62513 \text{ rad}$$

$$\approx 35.82^\circ$$

(6)

09-39



ε gefið með θ og m

finna v og θ'

samadæmi og dæmið að undan, og nú sést hvers vegna ekki var gott að setja strax inn tölur þar

$$v = v_0\sqrt{\epsilon^2\cos^2\theta + \sin^2\theta'}$$

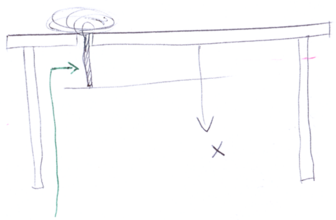
$$\theta' = \arctan\left\{\frac{1}{\epsilon}\tan\theta\right\}$$

(7)

09-15

Klukkan t=0 fellur reipið niður um gatið. Ekkert viðnám, finna v og a sem föll af x

Heildarlengd reipis er L



$$\rho = \frac{M}{L}$$

$m = \rho x$ er massi reipis sem hangir

$$\frac{dp}{dt} = mg = \rho g x$$

$$\frac{d}{dt}(mv) = \dot{m}v + m\dot{v} = \rho\dot{x}v + m\dot{v} = \rho v^2 + m\dot{v}$$

$$\rho v^2 + \rho x\dot{v} = \rho g x \quad \to \quad v^2 + x\dot{v} = gx$$

$$\to \quad v^2 + x\frac{dv}{dt} = gx \quad \to \quad v^2 + x\frac{dv}{dx}\frac{dx}{dt} = gx$$

(8)

$$\boxed{v^2 + xv\frac{dv}{dx} = gx}$$

ólínuleg diffurjafna fyrir v sem falli af x (t er horfið)

upphafsskilyrði $v(0) = 0$, $x(0) = 0$, reynum því veldislausn

$v = Cx^\alpha$, setjum inn í jöfnu

$$(Cx^\alpha)^2 + x(Cx^\alpha)(C\alpha x^{\alpha-1}) = gx$$

$$\to \quad C^2x^{2\alpha} + C^2\alpha x^{2\alpha} = gx$$

$$C^2\{1+\alpha\}x^{2\alpha} = gx$$

$$C^2\{1+\alpha\}x^{2\alpha} = gx$$

því fæst $2\alpha = 1$

og

$$\underbrace{C^2\{1+\alpha\} = g}$$

$$C^2\frac{3}{2} = g$$

$$\to \quad C = \sqrt{\frac{2g}{3}}$$

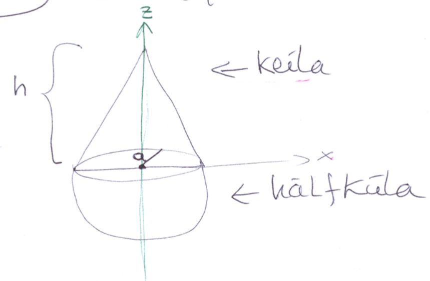
því er lausnin

$$\boxed{v = \sqrt{\frac{2gx}{3}}}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \sqrt{\frac{2gx}{3}}\,\frac{1}{2}\sqrt{\frac{2g}{3x}}$$

$$= \frac{2g}{3}\cdot\frac{1}{2} = \frac{g}{3}$$

09-03 CM fyrir



samhverft um z-ás

→ $y_{cm} = 0$, $x_{cm} = 0$

Verðum að finna z_{cm}

Sívalningur:

$$z_{cm} = \frac{1}{M}\int_A z\,dm$$

$$z_{cm} = \frac{\int_0^{2\pi} d\theta \int_0^a r\,dr \int_0^{h-\frac{hr}{a}} dz\, \rho z}{\int_0^{2\pi} d\theta \int_0^a r\,dr \int_0^{h-\frac{hr}{a}} dz\, \rho}$$

$$z_{cm} = \frac{\frac{h^2}{2}\int_0^a r\,dr\left(1-\frac{r}{a}\right)^2}{h\int_0^a r\,dr\left(1-\frac{r}{a}\right)} = \frac{\frac{h^2}{2}\,\frac{a^2}{12}}{h\,\frac{a^2}{6}} = h\cdot\frac{1}{4}$$

Hálfkúla:

$z = r\cos\theta$

$$z_{cm} = \frac{\int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} \sin\theta\, d\theta \int_0^a r^2 dr\, \rho z}{\int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} \sin\theta\, d\theta \int_0^a r^2 dr\, \rho}$$

$$z_{cm} = \frac{2\pi\int_{\pi/2}^{\pi} \sin\theta\cos\theta\cdot d\theta \int_0^a r^3 dr\cdot\rho}{\frac{4\pi}{6}a^3\cdot\rho} = \frac{-\frac{1}{2}\cdot\frac{a^4}{4}\rho}{\frac{1}{3}a^3\cdot\rho}$$

$$= -\frac{3}{8}a$$

→ Ef hlutirnir eru með sama massaþéttleika fast

$$z_{cm} = \frac{m_k (z_{cm})_k + m_{hk}(z_{cm})_{hk}}{m_k + m_{hk}}$$

(13)

$$z_{cm} = \frac{\left(2\pi \frac{ha^2}{6}\varrho\right)\cdot\frac{h}{4} - \left(\frac{4\pi}{6}a^3\varrho\right)\frac{3a}{8}}{2\pi\frac{ha^2}{6}\varrho + \frac{4\pi}{6}a^3\varrho}$$

$$= \frac{h^2 - 3a^3}{4(h+2a)}$$

(1)

10-12 Á breiddargráðu λ reikna frávik lóðlínu frá „lóðréttu" skilgreindu án snúnings. Sýna að

$$\epsilon = \frac{R\omega^2 \sin\lambda\cos\lambda}{g_0 - R\omega^2\cos^2\lambda}$$

Notum (10.32)

$$\bar F_{eff} = \bar S + m\bar g_0 - m\bar\omega\times(\bar\omega\times\bar r) - 2m\bar\omega\times\bar v_r$$

Setjum

$\bar S = 0$ aðrir kraftar

$\bar v_r = 0$ enginn hraði lóðs

$\bar r = (0,0,R)$ geisli jarðar

$\bar\omega = (-\omega\cos\lambda, 0, \omega\sin\lambda) \leftarrow \bar\omega$ í local hnitum r

(2)

$$\bar g_0 = (0,0,-g_0)$$

því fæst að

$$\bar\omega\times\bar r = (-\omega\cos\lambda, 0, \omega\sin\lambda)\times(0,0,R)$$
$$= \omega R\cos\lambda\cdot\hat e_{y_r}$$

$$-m\bar\omega\times(\bar\omega\times\bar r) = -m(-\omega\cos\lambda, 0, \omega\sin\lambda)\times(0,1,0)\,\omega R\cos\lambda$$
$$= m\omega^2 R\left\{\sin\lambda\cos\lambda\cdot\hat e_{x_r} + \cos^2\lambda\cdot\hat e_{z_r}\right\}$$

og því

$$\bar F_{eff} = -mg_0\hat e_{z_r} + m\omega^2 R\left\{\sin\lambda\cos\lambda\,\hat e_{x_r} + \cos^2\lambda\,\hat e_{z_r}\right\}$$

Sem hefur bæði $\hat e_{x_r}$ og $\hat e_{z_r}$ þætti

(3)

Því gildir um frávikið að

$$\tan\epsilon = \frac{\left|(F_{eff})_{x_r}\right|}{\left|(F_{eff})_{z_r}\right|} = \frac{\omega^2 R\sin\lambda\cos\lambda}{g_0 - \omega^2 R\cos^2\lambda}$$

$$= \frac{\sin\lambda\cos\lambda}{\frac{g_0}{\omega^2 R} - \cos^2\lambda}$$

$$\frac{g_0}{\omega^2 R} = \frac{9.81\ \mathrm{m/s^2}}{\left(7.3\cdot 10^{-5}\frac{1}{s}\right)^2 6.4\cdot 10^5\,\mathrm{m}} \sim 2.9\cdot 10^3$$

$$\rightarrow \epsilon \approx \frac{\sin\lambda\cos\lambda}{\frac{g_0}{\omega^2 R} - \cos^2\lambda} \approx \frac{\sin\lambda\cos\lambda}{\frac{g_0}{\omega^2 R}}$$
$$= \frac{\omega^2 R}{g_0 2}\sin(2\lambda)$$

(10-03) μ_s , $\bar{r}' = \bar{R} + \bar{r}$ (4)

fast kerfi

snúnings kerfi

upphaf snúningskerfis

Jafna (10.25) gefur

í fasta kerfinu

$$\bar{F}_{eff} = \bar{F} - m\ddot{\bar{R}}_f - m\dot{\bar{\omega}} \times \bar{r} - m\bar{\omega} \times (\bar{\omega} \times \bar{r}) - 2m\bar{\omega} \times \bar{v}_r$$

↑ í snúnings kerfinu

Einu kraftarnir eru miðsóknar og viðnáms
Hversu langt frá miðju getur pökkurinn legið án
þess að renna til

$$\rightarrow \quad \mu_s mg = m\omega^2 r \quad \rightarrow \quad r = \frac{\mu_s g}{\omega^2}$$

(10-05) Eins og ne... er í Ex. 10.2 í bókinni (5)

$$\bar{F}_{eff} = m\bar{\omega} \times (\bar{\omega} \times \bar{r}) - 2m\bar{\omega} \times \bar{v}_r$$

$$\bar{a}_{eff} = -\bar{\omega} \times (\bar{\omega} \times \bar{r}) - 2\bar{\omega} \times \bar{v}_r$$

$$\bar{\omega} = \omega \hat{e}_z \qquad \bar{\omega} = (0,0,\omega)$$

$$\bar{r} = (x, y, 0)$$

$$\bar{v}_r = (\dot{x}, \dot{y}, \dot{z})$$

$$\rightarrow \quad \bar{a}_{eff} = (\ddot{x}, \ddot{y}, 0) = \omega^2 (x, y, 0) - 2\omega(\dot{x}, -\dot{y}, 0)$$

$$\rightarrow \quad \boxed{\begin{aligned} \ddot{x} &= \omega^2 x + 2\omega \dot{y} \\ \ddot{y} &= \omega^2 y - 2\omega \dot{x} \end{aligned}}$$

Umskrifum sem (6)

$$y_1 = x \quad \rightarrow \quad \dot{y}_1 = \dot{x} = y_2$$

$$y_2 = \dot{x} \quad \rightarrow \quad \dot{y}_2 = \ddot{x} = \omega^2 x + 2\omega \dot{y} = \omega^2 y_1 + 2\omega y_4$$

$$y_3 = y \quad \rightarrow \quad \dot{y}_3 = \dot{y} = y_4$$

$$y_4 = \dot{y} \quad \rightarrow \quad \dot{y}_4 = \ddot{y} = \omega^2 y - 2\omega \dot{x} = \omega^2 y_3 - 2\omega y_2$$

Kneppið er því

$$\boxed{\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \omega^2 y_1 + 2\omega y_4 \\ \dot{y}_3 &= y_4 \\ \dot{y}_4 &= \omega^2 y_3 - 2\omega y_2 \end{aligned}}$$

upphafsskilyrði

$$y_1 = -0{,}5 \text{ m}$$

$$y_3 = 0 \text{ m}$$

$$y_2 = \frac{v_0}{\sqrt{2}} \ \frac{\text{m}}{\text{s}}$$

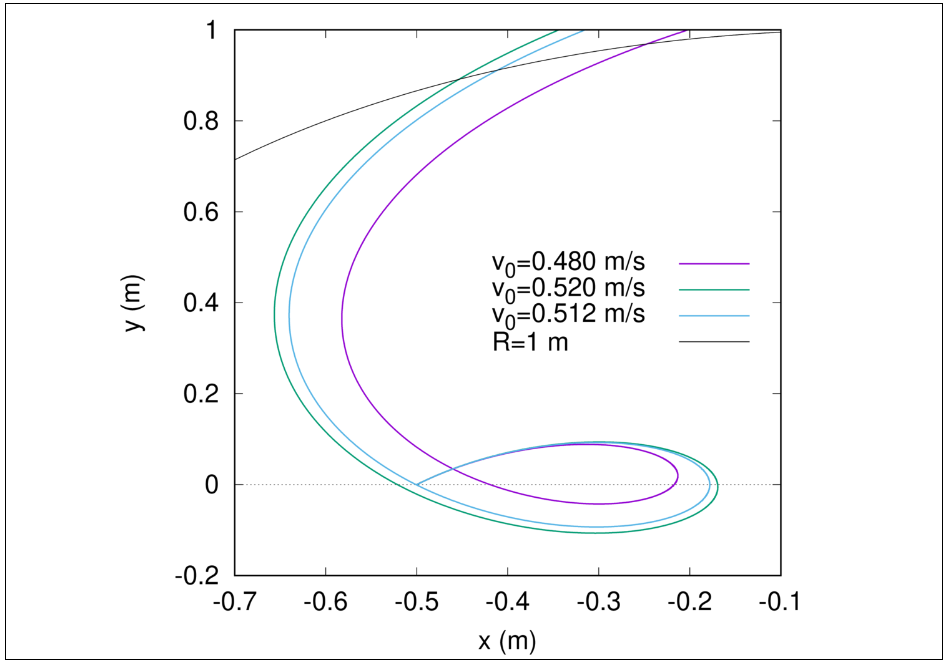
$$y_4 = \frac{v_0}{\sqrt{2}} \ \frac{\text{m}}{\text{s}}$$

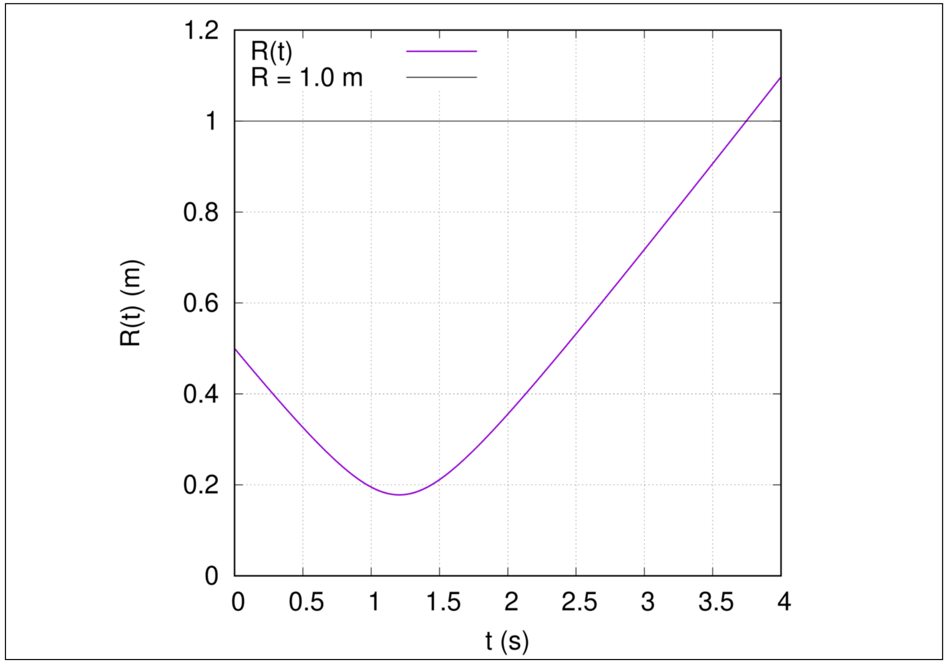
Samkvæmt Ex 10.2 þarf að regna v_0 á bilinu (7)

$$0{,}47 \ \frac{\text{m}}{\text{s}} < v_0 < 0{,}53 \ \frac{\text{m}}{\text{s}} \ , \qquad \omega = 1 \ \frac{\text{rad}}{\text{s}} \ , \quad R = 1 \text{ m}$$

Viljum finna v_0 sem fær pökkinn til að koma aftur í gegnum upphafspunktinn, og tíman þar til hann fer út af

Sjá gröf á tveimur næstu síðum





10-15

10

Höfum fasta kerfið með hnit x_f, og snúningskerfið með x_r

Gerum ráð fyrir að upphafspunktar kerfanna falli saman

$$\to \quad \bar{v}_f = \bar{v}_r + \bar{\omega} \times \bar{r}_r$$

og í fasta kerfinu er

$$L = \frac{m}{2} v_f^2 - U(r_f)$$

með breyttum hnitum fæst

$$L = \frac{m}{2}\left\{ (\bar{v}_r + \bar{\omega} \times \bar{r}_r)^2 \right\} - U(r_r)$$

11

$$\to L = \frac{m}{2}\left\{ v_r^2 + 2\,\bar{v}_r \cdot (\bar{\omega} \times \bar{r}_r) + (\bar{\omega} \times \bar{r}_r)^2 \right\} - U(r_r)$$

$\bar{p}_r$ verðum við að skilgreina sem

$$\bar{p}_r = \frac{\partial L}{\partial \bar{v}_r} = m\bar{v}_r + m(\bar{\omega} \times \bar{r}_r)$$

og fall Hamiltons verður

$$H = \bar{v}_r \cdot \bar{p}_r - L = \frac{m}{2} v_r^2 - \frac{m}{2} (\bar{\omega} \times \bar{r}_r)^2 - U(r_r)$$

H er ekki fall af t, ekki heldur L

$$\to \frac{\partial H}{\partial t} = 0 \qquad \frac{\partial L}{\partial t} = 0$$

12

En hnitin $\bar r_f$ og $\bar r_r$ tengjast á tímaháðan hátt

$\longrightarrow H \neq E$

Skoðum liðinn

$$U_c = -\frac{m}{2}\left(\bar\omega\times\bar r_r\right)^2$$

$$(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d)-(a\cdot d)(b\cdot c)$$

$$(a\times b)\cdot(a\times b) = (a^2b^2-(a\cdot b)(a\cdot b)$$

$$-\nabla_r U_c = \frac{m}{2}\nabla_r\left\{\omega^2 r_r^2-(\bar\omega\cdot\bar r_r)^2\right\}$$

$$= m\left\{\omega^2\bar r_r-(\bar\omega\cdot\bar r_r)\bar\omega\right\} =$$

$$\bar\nabla(\bar a\cdot\bar b) = (\bar a\cdot\bar\nabla)\bar b+(\bar b\cdot\bar\nabla)a+\bar a\times(\bar\nabla\times\bar b)+\bar b\times(\bar\nabla\times\bar a)$$

$$\bar\nabla\times\bar r = 0$$

$$= \underbrace{-m\bar\omega\times(\bar\omega\times\bar r_r)}_{\text{miðsóknarkraftinum}}$$

$$\bar a\times(\bar b\times\bar c) = (\bar a\cdot\bar c)b-(\bar a\cdot\bar b)\bar c$$

13

10-22

Við $\lambda = 42^\circ N$ fellur blý $h = 27m$
reikna frávik frá lóðréttu ....

Coriolis hröðun $\bar a_c = 2\bar v\times\bar\omega$

$\longrightarrow$ frávik í austur með $|\bar a_c| = a_c = 2v\omega\cos\lambda$

$$\longrightarrow v_c(t) = \int_0^t a_c\,dt' = \int_0^t dt'\,2v(t')\omega\cos\lambda$$

$$= 2\omega\cos\lambda\int_0^t dt'\,(gt) = \omega\cos\lambda\cdot gt^2$$

$$\longrightarrow x_c(t) = \int_0^t dt'\,v_c(t') = \omega\cos\lambda\,\frac{gt^3}{3}$$

14

Falltími blýsins er $t = \sqrt{\frac{2h}{g}}$

$$\longrightarrow x_c(t) = \omega\cos\lambda\,\frac{g}{3}\left(\frac{2h}{g}\right)^{3/2}$$

$$= \omega\cos\lambda\cdot\frac{1}{3}\sqrt{\frac{8h^3}{g}}$$

$$= \left(7.3\cdot10^{-5}\,\tfrac{1}{s}\right)\cos(42^\circ)\sqrt{\frac{8(27m)^3}{9{,}81\,m/s^2}}$$

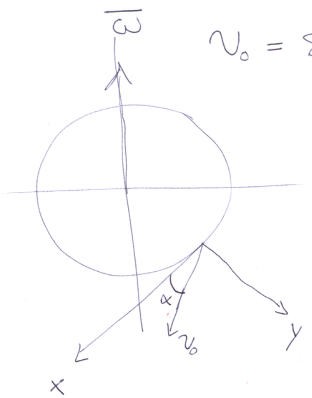
$$\approx 0{,}0023\,m$$

15

10-18

$\lambda = 50^\circ S$ skotið suður $\alpha = 37^\circ$

$v_0 = 800$ m/s hve stór er Coriolis geigun



$$v = \begin{pmatrix} v_x\\ v_y\\ 0\end{pmatrix} = \begin{pmatrix} v_0\cos\alpha\\ v_0\sin\alpha-gt\\ 0\end{pmatrix}$$

$$\bar\omega = \begin{pmatrix} -\omega\cos\lambda\\ -\omega\sin\lambda\\ 0\end{pmatrix}$$

$$\bar a_c = 2\bar v\times\bar\omega$$

$$= \hat e_z\left\{-2v_0\omega\cos\alpha\sin\lambda+2(v_0\sin\alpha-gt)\omega\cos\lambda\right\}$$

(16)

$$v_c = \int_0^t a_c(t')\,dt' = \hat{e}_z\left\{2v_0\omega t(\sin\alpha\cos\lambda - \cos\alpha\sin\lambda) - \omega\cos\lambda\cdot g t^2\right\}$$

flugtíminn er $t = \frac{2v_0\sin\alpha}{g}$

Geigunin er

$$z_c = \int_0^t v_c\,dt' = v_0\omega t^2(\sin\alpha\cos\lambda - \cos\alpha\sin\lambda) - \omega\cos\lambda\,\frac{g t^3}{3}$$

$\rightarrow z_c = -272$ m   inn í blaðið til austurs

(1)

11-12 Sýna að engin hverfitregða um höfuðás geti verið meiri en summa hinna tveggja

Hugsum um I_i

$$I_j + I_k = \int dv\{x_i^2 + x_k^2\}\rho + \int dv\{x_i^2 + x_j^2\}\rho$$

þar sem við höfum notað (11.15)

$$I_{ij} = \int_V dv\,\rho(r)\left\{\delta_{ij}\sum_k x_k^2 - x_i x_j\right\}$$

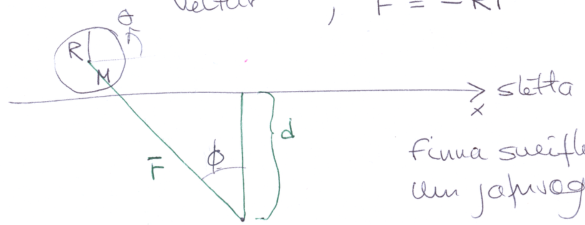
og $I_i = I_{ii}, \dots$

$$I_j + I_k = \int dv\{x_j^2 + x_k^2\}\rho + 2\int dv\,x_i^2\rho = I_i + 2\underbrace{\int dv\,x_i^2\rho}_{>0}$$

(2)

því fæst að $I_i \le I_j + I_k$

11-07 Veltur, $\bar{F} = -k\bar{r}$



Finna sveiflutíma dósts um jafnvægispunktinn

Velta eftir x-ás → kraftverkun á CM stefnu er

$F_x = -kr\sin\phi = -kx$, $I = \frac{MR^2}{2}$, $\rightarrow U = \frac{kx^2}{2}$

Hreyfiorkan er

$$T = \frac{M}{2}\dot{x}^2 + \frac{I}{2}\dot{\theta}^2 = \frac{M}{2}\dot{x}^2 + \frac{M(R\dot{\theta})^2}{4}$$

En $R\theta = x \rightarrow T = \frac{3M}{4}\dot{x}^2$

(3)

$$L = T - U = \frac{3M}{4}\dot{x}^2 - \frac{kx^2}{2}$$

Notum jöfnu Lagrange

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \rightarrow -kx - \frac{d}{dt}\left(\frac{3M}{2}\dot{x}\right) = 0$$

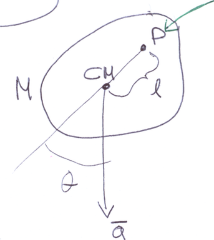
$\rightarrow \frac{3}{2}M\ddot{x} + kx = 0$ eða $\ddot{x} + \frac{2k}{3M}x = 0$

og því $\omega = \sqrt{\frac{2k}{3M}}$

11-15

4

upphengipunktur



Setning Steiners (11.49)

$$\to I = MR_0^2 + Ml^2$$

þekkjum ekki endilega R_0

$$T = \frac{I}{2}\dot\theta^2 , \quad U = Mgl(1-\cos\theta)$$

$$\to L = \frac{I}{2}\dot\theta^2 - Mgl(1-\cos\theta)$$

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = 0 \to -Mgl\sin\theta - \frac{I}{2}2\ddot\theta = 0$$

< litil sveifla

$$\to \ddot\theta + \underbrace{\frac{Mgl}{I}}\theta = 0$$

$$= \frac{Mgl}{MR_0^2+Ml^2} = \frac{gl}{R_0^2+l^2} = \omega^2$$

5

Ef við finnum annan punkt fyrir upphengju með sama ω og með l' þá gildir

$$\frac{gl}{R_0^2+l^2} = \frac{gl'}{R_0^2+(l')^2}$$

$$\to l\{R_0^2+(l')^2\} = l'\{R_0^2+l^2\}$$

$$\to R_0^2\{l-l'\} = l^2l' - (l')^2l = ll'\{l-l'\} \to \underline{R_0^2 = ll'}$$

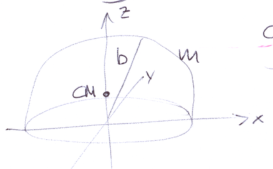
$$\to \omega = \sqrt{\frac{gl}{R_0^2+l^2}} = \sqrt{\frac{gl}{ll'+l^2}} = \sqrt{\frac{g}{l'+l}}$$

þarf ekki að mæla I, finna þarf tvo punkta með sömu tíðni

11-14

6

Gegnheilt hálfhvel, finna hverfitregðu höfuðása, og höfuðása m.v. CM



Kartískt hnit í gegnum miðju kúlunnar eru (x,y,z). Miðað við þau er hverfitregðufylkið J

fyrir CM vitum við að $x_{cm} = 0$, $y_{cm} = 0$

$$\rho = \frac{m}{V} = \frac{m}{\left(\frac{4}{3}\pi b^3\right)\frac{1}{2}} = \frac{3m}{2\pi b^3} \quad \text{fasti}$$

$r\cos\theta = z$

$$z_{cm} = \frac{1}{m}\int dv\,\rho z = \frac{\rho}{m}2\pi\int_0^{\pi/2}\sin\theta d\theta \cos\theta\int_0^b r^2dr\,r$$

$$= \frac{3}{2\pi b^3}2\pi\left(\frac{1}{2}\right)\frac{b^4}{4} = \frac{3b}{8}$$

7

Samhverfa $J_{11} = J_{22}$

$$J_{11} = \rho\int dv\{y^2+z^2\}$$

$$= \rho\int_0^b r^2dr\int_0^{\pi/2}d\theta\cdot\sin\theta\int_0^{2\pi}d\phi\left\{\sin^2\theta\sin^2\phi + \cos^2\theta\right\}r^2$$

$$= \frac{3m}{2\pi b^3}\frac{b^5}{5}\int_0^{\pi/2}d\theta\sin\theta\left\{\pi\sin^2\theta + 2\pi\cos^2\theta\right\}$$

$$= \frac{3m}{2\pi b^3}\frac{b^5}{5}\left\{\pi\cdot\frac{2}{3} + 2\pi\cdot\frac{1}{3}\right\} = \frac{mb^2}{2\cdot 5}\{2+2\}$$

$$= \frac{2mb^2}{5} \quad \text{sem er líka } J_{22}$$

$$J_{33} = \rho\int dv\{x^2+y^2\} = \rho\int_0^b r^4 dr\int_0^{2\pi} d\phi\int_0^{\pi/2}\sin\theta\, d\theta$$

$$\cdot\Big\{\sin^2\theta\big(\underbrace{\sin^2\phi+\cos^2\phi}_{=1}\big)\Big\}$$

$$= \frac{3m}{2\pi b^3}\frac{b^5}{5}2\pi\int_0^{\pi/2} d\theta\,\sin^3\theta = \frac{3m}{2\pi b^3}\frac{b^5}{5}2\pi\cdot\frac{2}{3}$$

$$= \frac{2mb^2}{5}$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$
$\Bigg\} \longrightarrow$ $I_{ij}=0$ ef $i\neq j$

I er á hornalínuham

Þurfum I ~~miðað~~ við CM

$$I_{33} = J_{33}$$

en fyrir hin er notuð setning Steiners

$$I_{11} = J_{11} - m z_{CM}^2 = J_{11} - m\left(\frac{3b}{8}\right)^2$$

$$= \frac{2}{5}mb^2 - m\frac{9b^2}{64} = \frac{83}{320}mb^2 = I_{22}$$

og áður var komið

$$I_{33} = \frac{2mb^2}{5}$$

(x',y',z')
Höfuðásarnir eru því hnitakerfið, kartíska, sem lyftir upp í z_{CM}, en ásarnir eru samsíða (x,y,z)

11-11

a) velta án þess að hann renni til

finna hornhraða þegar kubbur „lendir"

① Hæð CM yfir sléttu $\frac{l}{\sqrt{2}}$

② — || — $\frac{l}{2}$

① Einungis stöðuorka $U_1 = mg\frac{l}{\sqrt{2}}$

② stöðuorka $U_2 = mg\frac{l}{2}$

Hreyfiorka $T_2 = \frac{m}{2}v_{CM}^2 + \frac{I}{2}\omega^2$

Af mynd sést að $v_{CM} = \frac{l}{\sqrt{2}}\omega$

Samkvæmt Ex. 11.5 í bók er I hér um CM þvert á einn flötinn $I = \frac{m}{6}l^2$, og Steiner leiðir þá til I' um ásinn í gegnum P því fæst frá orku varðveislu

$$I' = I + m\left(\frac{l}{\sqrt{2}}\right)^2 = \frac{2}{3}ml^2$$

$$U_1 = U_2 + T_2$$

eða

$$mg\frac{l}{\sqrt{2}} = mg\frac{l}{2} + 0 + \frac{2m}{6}l^2\omega^2$$

Sleppum hliðrun massamiðju, einungis snúningur um P

$$\to\ gl\left\{\frac{1}{\sqrt{2}} - \frac{1}{2}\right\} = (\omega l)^2\left\{0 + \frac{1}{3}\right\} = (\omega l)^2\frac{1}{3}$$

$$gl\{\sqrt{2}-1\} = (\omega l)^2\frac{2}{3} \ \to\ \omega^2 = \frac{3g}{2l}\{\sqrt{2}-1\}$$

b) Enginn núningskraftur

Enginn láréttur kraftur
CM fellur lóðrétt
Í fallinu snýst
kubburinn um θ frá 0 í $\frac{\pi}{4}$

$\rightarrow$ $y_{CM} = \frac{l}{\sqrt{2}}\cos\theta$ (Hreyfingin er ekki með jöfnum hornhraða?)

$$\dot{y}_{CM} = -\frac{l}{\sqrt{2}}\dot{\theta}\sin\theta$$

og í lokapunktinum gildir $(\dot{y}_{CM})_2 = -\frac{1}{2}l\dot{\theta} = -\frac{l}{2}\omega$

þegar $\theta = \pi/4$

Orkuvarðveislan er þá

$$U_1 = U_2 + T_2$$

$$mg\frac{l}{\sqrt{2}} = \frac{mgl}{2} + \frac{m}{2}(\dot{y})_2^2 + \frac{I}{2}\omega^2$$

muna I um miðjuna

$$= \frac{mgl}{2} + \frac{m}{2}\left(-\frac{l\omega}{2}\right)^2 + \frac{ml^2}{12}\omega^2$$

$$\rightarrow\quad gl\left\{\frac{1}{\sqrt{2}} - \frac{1}{2}\right\} = (\omega l)^2\left\{\frac{1}{8} + \frac{1}{12}\right\}$$

eða

$$gl\left\{\sqrt{2} - 1\right\} = (\omega l)^2\left\{\frac{1}{4} + \frac{1}{6}\right\} = (\omega l)^2\frac{5}{12}$$

$$\rightarrow\quad \omega^2 = \frac{12}{5}\frac{g}{l}\left\{\sqrt{2} - 1\right\}$$

11-13

Þrír massar í hnitum

finna $\mathbb{I}$, höfuðása og höfuðhverfitregður

$m_1 = 3m$ í $(b,0,b)$
$m_2 = 4m$ í $(b,b,-b)$
$m_3 = 2m$ í $(-b,b,0)$

$$\mathbb{I}_{11} = \sum_\alpha m_\alpha\left\{x_{\alpha 2}^2 + x_{\alpha 3}^2\right\} = 3mb^2 + 4m(2b^2) + 2mb^2$$
$$= 13mb^2$$

$\mathbb{I}_{22} = 16mb^2$, $\mathbb{I}_{33} = 15mb^2$

$$\mathbb{I}_{12} = \mathbb{I}_{21} = -\sum_\alpha m_\alpha x_{\alpha 1}x_{\alpha 2} = -4mb^2 - 2m(-b^2) = -2mb^2$$

$\mathbb{I}_{13} = \mathbb{I}_{31} = mb^2$, $\mathbb{I}_{23} = \mathbb{I}_{32} = 4mb^2$

$$\mathbb{I} = mb^2\begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix}$$

Eigingildi

$17 - \sqrt{7} \approx 14.354\,mb^2$
$17 + \sqrt{7} \simeq 19.646\,mb^2$
$10 = 10.000\,mb^2$

höfuðhverfi-tregðar

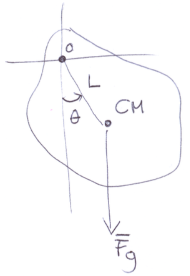
Eiginvigrar

$(1,1,-1)$
$(1,-\sqrt{7}-3,-\sqrt{7}-2)$
$(1,\sqrt{7}-3,\sqrt{2}-2)$

Höfuðásar

ekki normaðir, en eru með vídd b

(11-32) leysa Ex. 11.2 þegar sveiflur eru ekki smáar (1)



$\theta(0) = 67°$

finna $\dot\theta(t)$ þegar $\theta = 1°$

$M = 340\,g$, $L = 13\,cm$, $k = 17\,cm$

$T = \frac{I}{2}\dot\theta^2$, $U = -MgL\cos\theta$

$I = Mk^2$

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = 0 \rightarrow -MgL\sin\theta - I\ddot\theta = 0$$

$$\rightarrow \ddot\theta + \frac{MgL}{Mk^2}\sin\theta = 0$$

eða

$$\boxed{\ddot\theta + \frac{gL}{k^2}\sin\theta = 0}$$

Umskrifum hreyfijöfnu (2)

$$\frac{d}{dt}(\dot\theta) = \frac{gL}{k^2}\frac{d(\cos\theta)}{d\theta} \rightarrow d\theta\frac{d}{dt}(\dot\theta) = \frac{gL}{k^2}d(\cos\theta)$$

$$\rightarrow \dot\theta\, d\dot\theta = \frac{gL}{k^2}d(\cos\theta)$$

Heildum

$$\int_{\dot\theta(0)=0}^{\dot\theta}\dot\theta' d\dot\theta' = \frac{gL}{k^2}\int_{\cos\theta_0}^{\cos\theta} d(\cos\theta)' \rightarrow \frac{\dot\theta^2}{2} = \frac{gL}{k^2}\left\{\cos\theta - \cos\theta_0\right\}$$

$$\rightarrow \boxed{\dot\theta = \sqrt{\frac{2gL}{k^2}(\cos\theta - \cos\theta_0)}}$$

Höfum séð áður að $\dot\theta$ og því líka lotan eru háð útslaginu

$\dot\theta = \dot\theta(\theta, \theta_0)$, $\dot\theta(1°, 67°) = \sqrt{\frac{2\cdot 9.81\cdot 0.13}{(0.17)^2}(\cos 1° - \cos 67°)} \simeq 7{,}33$

(11-27) x_3 er samhverfuás hlutar, engir kraftar ~~...~~ vogi (3)

L hverfiþungi um x_3'

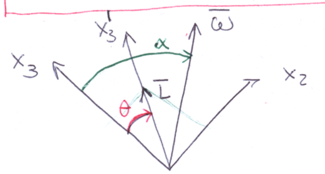
$I_1 = I_2$

Hornið milli $\bar\omega$ og x_3 er α

$\bar\omega$ og L eru upphaflega í x_2-x_3-sléttu

Hver er hornhraði x_3 um L (m.v. I_1, I_3, ω, α) ?

$\rightarrow \dot\phi$?



$L_1 = I_1\omega_1 = 0$

$L_2 = I_2\omega_2 = I_1\omega_2 = L\sin\theta = I_1\omega\sin\alpha$

$L_3 = I_3\omega_3 = L\cos\theta = I_3\omega\cos\alpha$

Lausn vogislausa snúðsins segir að L sé í sléttu (4)

$\bar\omega$ og $\hat e_3$ $\rightarrow$ $\hat e_3$ og $\bar\omega$ velta um $L \parallel \hat e_3'$

þegar $\hat e_2$ er í sléttu $\hat e_3, \bar\omega$ og L $\leftarrow$ upphafsgildi

$\rightarrow$ $\psi = 0$ og (11.102) gefur $\omega_2 = \dot\phi\sin\theta$

$$\rightarrow \dot\phi = \frac{\omega_2}{\sin\theta} = \frac{\omega\sin\alpha}{\sin\theta}$$

(sem er líka jafna (11.147) í bók)

En við höfum frá síðu (3)

$$\tan\theta = \frac{L_2}{L_3} = \frac{I_1}{I_3}\tan\alpha$$

$\omega_3 = \omega\cos\alpha$

Og hornafræðin

$$\sin\theta = \pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\rightarrow \dot\phi = \frac{\omega\sin\alpha}{\sin\theta} = \pm\,\omega\sin\alpha\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$$

$$\dot{\phi} = \pm\,\omega \sin\alpha \,\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$$

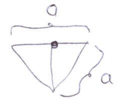
$$= \pm\,\omega \sin\alpha \,\frac{\sqrt{1+\left(\frac{I_1}{I_3}\right)^2\tan^2\alpha}}{\frac{I_1}{I_3}\tan\alpha}$$

$$= \pm\,\omega \sin\alpha \,\frac{I_3}{I_1}\sqrt{\cot^2\alpha + \left(\frac{I_1}{I_3}\right)^2}$$

$$= \pm\,\omega \,\frac{I_3}{I_1}\sqrt{\cos^2\alpha + \left(\frac{I_1}{I_3}\right)^2\sin^2\alpha}$$

$$= \pm\,\frac{\omega}{I_1}\sqrt{I_3^2\cos^2\alpha + I_1^2\sin^2\alpha}$$

11-24 Jafnarma þríhyrningur hengdur upp:

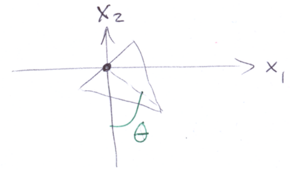
a) 

b) 

Hliðlengd a

↳ hæð h: $\sqrt{\frac{a^2}{4}+h^2} = a \;\to\; h = \sqrt{a^2 - \frac{a^2}{4}} = \frac{\sqrt{3}\,a}{2}$

flötur: $A = \frac{a}{2}\cdot h = \frac{\sqrt{3}\,a^2}{4} \;\to\; \rho = \frac{m}{A} = \frac{4m}{\sqrt{3}\,a^2}$

a) 

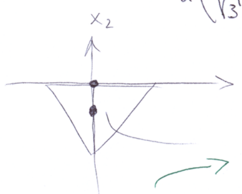
$$I_3 = \rho\int dx\,dy\,(x^2+y^2)$$

$$= 2\rho\int_0^{a/2} dx \int_{-\frac{\sqrt{3}}{2}(a-2x)}^{0} dy\,(x^2+y^2)$$

$$I_3 = 2\rho\int_0^{a/2} dx\left\{\left(x^2y + \frac{y^3}{3}\right)\Big|_{-\frac{\sqrt{3}}{2}(a-2x)}^{0}\right\}$$

$$= 2\rho\int_0^{a/2} dx\left\{x^2\cdot\frac{\sqrt{3}}{2}(a-2x) + \frac{1}{3}\left(\frac{\sqrt{3}}{2}(a-2x)\right)^3\right\} = 2\rho\,\frac{a^4}{16\sqrt{3}}$$

$$= 2\left(\frac{4m}{\sqrt{3}\,a^2}\right)\frac{a^4}{16\sqrt{3}} = \frac{m}{6}\,a^2$$



þegar $\theta = 0$

$$Y_{cm} = \frac{2\rho}{m}\int_0^{a/2} dx \int_{-\frac{\sqrt{3}}{2}(a-2x)}^{0} y\,dy$$

$$= -\frac{2\rho}{m}\int_0^{a/2} dx\left\{\frac{1}{2}\left(\frac{\sqrt{3}}{2}(a-2x)\right)^2\right\} = -\frac{\rho}{m}\,\frac{a^3}{8}$$

$$Y_{cm} = -\frac{\rho}{m}\,\frac{a^3}{8} = -\frac{4}{\sqrt{3}\,a^2}\,\frac{a^3}{8} = -\frac{a}{2\sqrt{3}}$$

Hreyfiorka: $T = \frac{I_3}{2}\dot{\theta}^2 = \frac{m}{12}\,a^2\dot{\theta}^2$ lengd sveifluáss

Stöðuorka $U = mg|Y_{cm}|(1-\cos\theta) = +\frac{mga}{2\sqrt{3}}(1-\cos\theta)$

$$\to\; L = T - U = \frac{m}{12}(a\dot{\theta})^2 - \frac{mga}{2\sqrt{3}}(1-\cos\theta)$$

því leiðir Euler-Lagrange til hreyfijöfnu

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = 0$$

$$\to\; -\frac{mga}{2\sqrt{3}}\sin\theta - \frac{ma^2}{6}\ddot{\theta} = 0$$

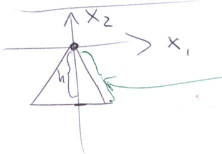
9

→ $\boxed{\ddot{\theta} + \sqrt{3}\,\frac{g}{a}\sin\theta = 0}$

Berum saman við einfaldan pendúl

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

$$\rightarrow \omega = \sqrt{\sqrt{3}\,\frac{g}{a}}$$



$$I_3' = 2\rho \int_{-\frac{\sqrt{3}a}{2}}^{0} dy \int_{0}^{-\frac{y}{\sqrt{3}}} dx\,(x^2+y^2)$$

$-h \rightarrow -\frac{\sqrt{3}a}{2}$

$$= 2\rho \int_{-\frac{\sqrt{3}a}{2}}^{0} dy \left\{ \left(\frac{x^3}{3} + xy^2\right)\Big|_{0}^{-y/\sqrt{3}} \right\} = -2\rho \int_{-\frac{\sqrt{3}a}{2}}^{0} dy \left\{ \frac{y^3}{3(\sqrt{3})^3} + \frac{y^3}{\sqrt{3}} \right\}$$

10

$$I_3' = 2\rho \left\{ \frac{\left(\frac{\sqrt{3}a}{2}\right)^4}{4\cdot 3(\sqrt{3})^3} + \frac{\left(\frac{\sqrt{3}a}{2}\right)^4}{4\cdot\sqrt{3}} \right\} = 2\rho a^4 \left\{ \frac{1}{12\cdot 16} + \frac{3}{4\cdot 16} \right\}\sqrt{3}$$

$$= 2\rho a^2 \sqrt{3}\,\frac{10}{12\cdot 16} = a^2 m\,\frac{10}{6\cdot 4} = \frac{5}{12} m a^2$$

og því

$$L = \frac{5}{24} m (a\dot{\theta})^2 - U$$

fyrir sem

áður

$$U = mg(1-\cos\theta)\cdot(h - y_{cm}) = mg\,\frac{a}{\sqrt{3}}(1-\cos\theta)$$

$$\rightarrow L = \frac{5}{24} m (a\dot{\theta})^2 - mg\,\frac{a}{\sqrt{3}}(1-\cos\theta)$$

$$\rightarrow \ddot{\theta} + \frac{12 g}{5\sqrt{3}a}\sin\theta = 0 \quad \rightarrow \omega = \sqrt{\frac{12}{5\sqrt{3}}\,\frac{g}{a}}$$

11

11-21 (11.54) – (11.61) í fylgja framsetningu

þannig var þetta athugað í fyrirlestri 18

Endurbætum $\bar{L} = \mathbb{I}\,\bar{\omega}$

fylki sem stendur fyrir þinum

vigrar

$\mathbb{I}$ sínum hnitakerfi

$$\boxed{\bar{L}' = \mathbb{I}'\bar{\omega}'}$$

$\bar{x}' = \lambda\bar{x}$

$\bar{x} = \lambda^t\bar{x}'$

$\lambda\lambda^t = 1$

$\lambda^t = \lambda^{-1}$

hornrétt fylki

$$\bar{L} = \lambda^t\bar{L}', \quad \bar{\omega} = \lambda^t\bar{\omega}'$$

$$\lambda^t\bar{L}' = \lambda^t\mathbb{I}'\lambda\bar{\omega}$$

$$\bar{L} = \underbrace{(\lambda^t\mathbb{I}'\lambda)}\,\bar{\omega} \rightarrow \mathbb{I} = \lambda^t\mathbb{I}'\lambda$$

og

$$\lambda\mathbb{I}\lambda^t = \mathbb{I}'$$

12

11-29 Samhverfur snúður þegar x_3' og x_3 eru í sömu átt

Sjá mynd 11-15 → $\theta = 0$

$$\left.\begin{aligned} \rightarrow p_\phi &= I_3\{\dot{\phi} + \dot{\psi}\} \\ p_\psi &= I_3\{\dot{\phi} + \dot{\psi}\} \end{aligned}\right\} = I_3\omega_3$$

og (11.159) →

fyrir orkuna fæst $(\theta = 0, \dot{\theta} = 0)$

$$E = \frac{I_3}{2}\omega_3^2 + Mgh$$

$$\rightarrow E' = E - \frac{I_3}{2}\omega_3^2 = Mgh$$

Hvað með stöðuleika snúðsins?

Notum (11.161)

$$E' = \frac{I_1}{2}\dot{\theta}^2 + \frac{\{P_\phi - P_\psi \cos\theta\}^2}{2I_1 \sin^2\theta} + Mgh\cos\theta = Mgh$$

$$= \frac{I_1}{2}\dot{\theta}^2 + \frac{(I_3\omega_3)^2\{1-\cos\theta\}^2}{2I_1 \sin^2\theta} + Mgh\cos = Mgh$$

Breytuskipti $z = \cos\theta$

$$\frac{d\cos\theta}{dt} = -\sin\theta\frac{d\theta}{dt} \rightarrow \frac{dz}{dt} = -\sin\theta\frac{d\theta}{dt}$$

$$\frac{I_1}{2}\frac{\dot{z}^2}{\sin^2\theta} + \frac{(I_3\omega_3)^2(1-z)^2}{2I_1\sin^2\theta} - Mgh\{1-z\} = 0$$

$$\frac{I_1}{2}\frac{\dot{z}^2}{(1-z^2)} + \frac{(I_3\omega_3)^2(1-z)^2}{2I_1(1-z^2)} - Mgh(1-z) = 0$$

$$\rightarrow \dot{z}^2 = \frac{(I_3\omega_3)^2(1-z)^2}{I_1^2} - \frac{2Mgh(1-z)(1-z^2)}{I_1}$$

$$= \frac{(1-z)^2}{I_1^2}\left\{2MghI_1(1+z) - (I_3\omega_3)^2\right\}$$

Þurfum $\dot{z}^2 \geq 0$, gerum ráð fyrir að snúður snúist hratt þ.a. $\{\cdots\} < 0 \rightarrow z = 1 \rightarrow \theta = 0$

$\rightarrow$ stöðug hreyfing ef $\rightarrow 4MghI_1 - (I_3\omega_3)^2 < 0$

$$\rightarrow \boxed{\frac{4MghI_1}{(I_3\omega_3)^2} < 1}$$

Síðan má stöðuga hegð gerist ef ω_3 er ekki stórt .....

(11-31) Þunn einsleit plata x - kerfi plötu

I_1, $I_2 > I_1$ um höfuðásana

$I_3 = I_1 + I_2$

O og O' falla saman við CM plötu

$t = 0$ sett á snúning í kraftfríu umhverfi með Ω um ás sem hallar α frá flöt sléttu og þvert á x_2-ás

Ef $\frac{I_1}{I_2} = \cos(2\alpha)$ sýna að

$$\omega_2(t) = \Omega\cos\alpha \cdot \tanh\{\Omega t\sin\alpha\}$$

$$I_1 = I_2\cos(2\alpha)$$

$$I_3 = I_1 + I_2 = I_2\{1+\cos(2\alpha)\} = 2I_2\cos^2\alpha$$

$$I_1 - I_2 = I_2\{\cos(2\alpha) - 1\} = -I_2\{1-\cos(2\alpha)\}$$

$$= -2I_2\sin^2\alpha$$

Kraftfrítt umhverfi (11.114)

$$\{I_2 - I_3\}\omega_2\omega_3 - I_1\dot{\omega}_1 = 0 \quad (1)$$

$$\{I_3 - I_1\}\omega_3\omega_1 - I_2\dot{\omega}_2 = 0 \quad (2)$$

$$\{I_1 - I_2\}\omega_1\omega_2 - I_3\dot{\omega}_3 = 0 \quad (3)$$

(17)

Notum upplýsingarnar um II

$$\left\{-2I_2 \sin^2\alpha\right\}\omega_1\omega_2 - 2I_2\cos^2\alpha\cdot\dot{\omega}_3 = 0 \quad (3)$$

$$I_2\left\{1-2\cos^2\alpha\right\}\omega_2\omega_3 - I_2\cos(2\alpha)\cdot\dot{\omega}_1 = 0 \quad (1)$$

$$I_2\left\{2\cos^2\alpha - \cos(2\alpha)\right\}\omega_3\omega_1 - I_2\dot{\omega}_2 = 0 \quad (2)$$

$$(3) \to \quad \dot{\omega}_3 = -\omega_1\omega_2\tan^2\alpha$$

$$(1) \to \quad \dot{\omega}_1 = -\omega_2\omega_3$$

$$(2) \to \quad \dot{\omega}_2 = \omega_3\omega_1$$

(18)

$$\to \omega_1\omega_2\omega_3 = \dot{\omega}_2\omega_2 \quad \leftarrow (2)$$

$$= -\dot{\omega}_1\omega_1 \quad \leftarrow (1)$$

$$= -\dot{\omega}_3\cdot\omega_3\cot^2\alpha \quad \leftarrow (3)$$

Heildun

$$\omega_2^2 - \omega_2^2(0) = -\omega_1^2 + \omega_1^2(0) = -\omega_3^2\cot^2\alpha + \omega_3^2(0)\cdot\cot^2\alpha$$

upphafsgildi

$$\omega_2(0) = 0 \quad \leftarrow \text{punkt á } x_2$$

$$\omega_1(0) = \Omega\cos\alpha$$

$$\omega_3(0) = \Omega\sin\alpha$$

(19)

þá fæst

$$\omega_2^2 = -\omega_1^2 + \Omega^2\cos^2\alpha = -\omega_3^2\cot^2\alpha + \Omega^2\cos^2\alpha$$

$$(2) \to \quad \dot{\omega}_2^2 = \omega_3^2\omega_1^2 \quad \text{og} \quad \omega_1^2 = \omega_3^2\cot^2\alpha$$

$$\dot{\omega}_2 = \omega_3^2\cot\alpha$$

$$\omega_3^2 = \Omega^2\sin^2\alpha - \omega_2^2\tan^2\alpha$$

$$\dot{\omega}_2 = -(\cot\alpha)\cdot\left\{\omega_2^2\tan^2\alpha - \Omega^2\sin^2\alpha\right\}$$

$$\to \frac{\dot{\omega}_2}{\left\{\omega_2^2\tan^2\alpha - \Omega^2\sin^2\alpha\right\}} = -\cot\alpha$$

(20)

Notum að $\dot{\omega}_2 = \frac{d\omega_2}{dt}$

$$\to \int_0^{\omega_2} \frac{d\omega_2'}{\omega_2'^2\tan^2\alpha - \Omega^2\sin^2\alpha} = \cot\alpha\int_0^t dt$$

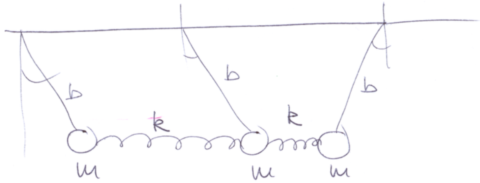
ef $\omega_2^2\tan^2\alpha < \Omega^2\sin^2\alpha$

$$(E.4c) \to \quad -\frac{1}{\tan\alpha\cdot\Omega\sin\alpha}\,\mathrm{Artanh}\left\{\frac{\omega_2\tan\alpha}{\Omega\sin\alpha}\right\} = -t\cos\alpha$$

$$\to \boxed{\omega_2(t) = \Omega\cos\alpha\cdot\tanh\left\{\Omega t\sin\alpha\right\}}$$

sem er rétt

12-26 (1)



$k = 0{,}2\ \frac{N}{m}$

$m = 0{,}25\ Kg$

$b = 0{,}47\ m$

$$T = \frac{m}{2}\sum_{i=1}^{3}(b\dot\theta_i)^2$$

$$U = mgb\left\{(1-\cos\theta_1)+(1-\cos\theta_2)+(1-\cos\theta_3)\right\} + \frac{kb^2}{2}\left[(\sin\theta_2-\sin\theta_1)^2+(\sin\theta_3-\sin\theta_2)^2\right]$$

$$\simeq \frac{mgb}{2}\left\{\theta_1^2+\theta_2^2+\theta_3^2\right\} + \frac{kb^2}{2}\left\{\theta_1^2+\theta_3^2+2\theta_2^2-2\theta_1\theta_2-2\theta_2\theta_3\right\}$$

(2)

$$M = \frac{mb^2}{2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$A = \frac{1}{2}\begin{pmatrix} mgb+kb^2 & -kb^2 & 0\\ -kb^2 & mgb+2kb^2 & -kb^2\\ 0 & -kb^2 & mgb+kb^2\end{pmatrix}$$

Þurfum að leysa eigingildisverkefnið

$$A\bar a = \omega^2 M \bar a$$

Hægt er að láta þáttinn $\frac{1}{2}$ í M og A falla niður

M er á hornalínu formi, því er eigingildisverkefnið venjulegt

(3)

$$A = \frac{1}{2}mgb\begin{pmatrix}1+\epsilon & -\epsilon & 0\\ -\epsilon & 1+2\epsilon & -\epsilon\\ 0 & -\epsilon & 1+\epsilon\end{pmatrix}, \quad \epsilon = \frac{kb^2}{mgb} = \frac{kb}{mg}$$

$$\longrightarrow \omega_i^2 = \begin{cases}\frac{mgb}{mb^2}\{1+3\epsilon\} & i=1\\ \frac{mgb}{mb^2}\{1\} & i=2\\ \frac{mgb}{mb^2}\{1+\epsilon\} & i=3\end{cases}$$

$$\rightarrow \omega_1 = \sqrt{\frac{g}{b}+\frac{3k}{m}} \qquad \omega_3 = \sqrt{\frac{g}{b}+\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{g}{b}}$$

(4)

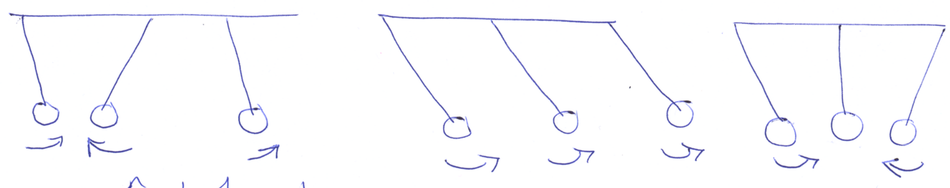
Eiginvigrar eru

$$\bar a = \begin{cases}\frac{1}{\sqrt6}(\theta_1, -2\theta_2, \theta_3) & i=1\\ \frac{1}{\sqrt3}(\theta_1,\theta_2,\theta_3) & i=2\\ \frac{1}{\sqrt2}(\theta_1, 0, -\theta_3) & i=3\end{cases}$$

með sveifluhætti

lægst orkan, minnst víxlverkan

1 2 3



← hæsta orkan

12-19 Þrír pendúlar með

$$U = \frac{1}{2}\left\{ \theta_1^2 + \theta_2^2 + \theta_3^2 - 2\epsilon_{12}\theta_1\theta_2 - 2\epsilon_{13}\theta_1\theta_3 - 2\epsilon_{23}\theta_2\theta_3 \right\}$$

Öll ϵ_{ij} mismunandi

Ekki bara nágrannar víxlverkast eins í dæminu að undan

Setjum

$$T = \frac{1}{2}\left\{ \dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2 \right\} \Rightarrow M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

og

$$A = \begin{Bmatrix} 1 & -\epsilon_{12} & -\epsilon_{13} \\ -\epsilon_{12} & 1 & -\epsilon_{23} \\ -\epsilon_{13} & -\epsilon_{23} & 1 \end{Bmatrix}$$

Sýna að margfeldni verði aðeins ef öll ϵ_{ij} eru þau sömu

Eigingildin má finna með wxmaxima, en þau eru flókin. Reynum því

$$\left\{ A - \omega^2 M \right\} \bar{a} = 0$$

línulegar óháðar jöfnur $\rightarrow \det\left\{ A - \omega^2 M \right\} = 0$

$$\rightarrow \begin{vmatrix} 1-\omega^2 & -\epsilon_{12} & -\epsilon_{13} \\ -\epsilon_{12} & 1-\omega^2 & -\epsilon_{23} \\ -\epsilon_{13} & -\epsilon_{23} & 1-\omega^2 \end{vmatrix} = 0$$

$$\rightarrow (1-\omega^2)^3 - (1-\omega^2)(\epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{23}^2) - 2\epsilon_{12}\epsilon_{13}\epsilon_{23} = 0$$

Setjum $1 - \omega^2 = x$

$$x^3 - x\cdot B - C = 0$$

$$\Delta = -4B^3 - 27C^2 = 4(\epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{23}^2)^3 - 108(\epsilon_{12}\epsilon_{13}\epsilon_{23})^2$$

Ef $\Delta > 0$ þá eru þrjár mismunandi rætur

$$(\epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{23}^2)^3 - 27(\epsilon_{12}\epsilon_{13}\epsilon_{23})^2 > 0$$

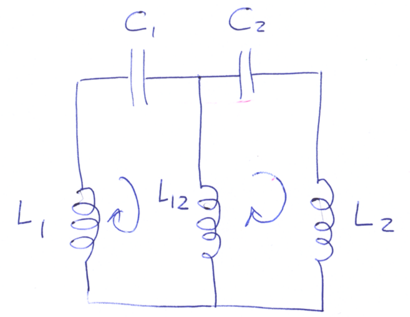
Ef $\Delta = 0$ þá er ein rót tvöföld

$\rightarrow$ ef

$$(\epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{23}^2)^3 = 27(\epsilon_{12}\epsilon_{13}\epsilon_{23})^2$$

gerist aðeins ~~ef~~ $\epsilon_{12} = \epsilon_{13} = \epsilon_{23}$

12-13



Lögmál Faradays

$$\frac{q}{C} = -L\frac{di}{dt}$$

(ekki Kirchhoffs)

$$L_1\frac{di_1}{dt} + \frac{q_1}{C_1} + L_{12}\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) = 0$$

$$L_2\frac{di_2}{dt} + \frac{q_2}{C_2} + L_{12}\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) = 0$$

Tvær rásir með einfaldri hreintóna sveiflu tengdar saman

9

Notum $\dot{q} = i$ $\left(\frac{dq}{dt} = i\right)$ og tökum tíma afleiðu

$$L_1 \frac{d^2 i_1}{dt^2} + \frac{i_1}{C_1} + L_{12}\left(\frac{d^2 i_1}{dt^2} - \frac{d^2 i_2}{dt^2}\right) = 0$$

$$L_2 \frac{d^2 i_2}{dt^2} + \frac{i_2}{C_2} + L_{12}\left(\frac{d^2 i_2}{dt^2} - \frac{d^2 i_1}{dt^2}\right) = 0$$

umritum

$$\{L_1 + L_{12}\}\frac{d^2 i_1}{dt^2} + \frac{i_1}{C_1} - L_{12}\frac{d^2 i_2}{dt^2} = 0$$

$$\{L_2 + L_{12}\}\frac{d^2 i_2}{dt^2} + \frac{i_2}{C_2} - L_{12}\frac{d^2 i_1}{dt^2} = 0$$

Gerum ráð fyrir lausnum

$$i_1(t) = Ae^{i\omega t}, \quad i_2(t) = Be^{i\omega t}$$

10

Þá fæst

$$\left[\omega^2\{L_1 + L_{12}\} - \frac{1}{C_1}\right]A - L_{12}\omega^2 B = 0$$

$$\left[\omega^2\{L_2 + L_{12}\} - \frac{1}{C_2}\right]B - L_{12}\omega^2 A = 0$$

$$\begin{pmatrix} \omega^2\{L_1+L_{12}\} - \frac{1}{C_1} & -L_{12}\omega^2 \\ -L_{12}\omega^2 & \omega^2\{L_2+L_{12}\} - \frac{1}{C_2} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

óhliðruð línuleg jafna $\rightarrow$ ákveðan af fylkinu verður að hverfa

11

$$\left\{\omega^2[L_1+L_{12}] - \frac{1}{C_1}\right\}\left\{\omega^2[L_2+L_{12}] - \frac{1}{C_2}\right\} - L_{12}^2\omega^4 = 0$$

$$\rightarrow \omega^2 = \frac{(L_1+L_{12})C_1 + (L_2+L_{12})C_2 \pm \sqrt{[(L_1+L_{12})C_1 - (L_2-L_{12})C_2]^2 - 4L_{12}^2C_1C_2}}{2C_1C_2\left[(L_1+L_{12})(L_2+L_{12}) - L_{12}^2\right]}$$

ef $L_{12} \rightarrow 0$, og $L_1 = L_2 = L$, $C_1 = C_2 = C$

$$\rightarrow \omega^2 = \frac{1}{LC}$$

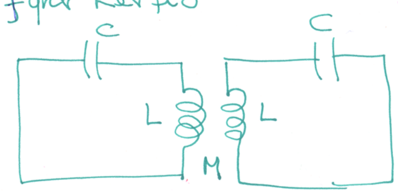
12

12-12 Höfum

$$L\ddot{I}_1 + \frac{I_1}{C} + M\ddot{I}_2 = 0$$

$$L\ddot{I}_2 + \frac{I_2}{C} + M\ddot{I}_1 = 0$$

fyrir Kerfið



notum þessa jöfnu og eins . . . .

$$L\ddot{I}_1 + \frac{I_1}{C} + M\left\{-\frac{I_2}{LC} - \frac{M}{L}\ddot{I}_1\right\} = 0$$

$$\rightarrow \left\{L - \frac{M^2}{L}\right\}\ddot{I}_1 + \frac{I_1}{C} - \frac{M}{LC}I_2 = 0$$

einsfæst

$$\left\{L - \frac{M^2}{L}\right\}\ddot{I}_2 + \frac{I_2}{C} - \frac{M}{LC}I_1 = 0$$

Tengingin er núna ekki um lið með $\ddot{I}_i$

(13)

Berum saman við (12.1) og (12.8) fyrir 2 massa tengda gorm

Setjum

$$m = L - \frac{M^2}{L}, \quad k_{12} = \frac{M}{LC}, \quad k = \frac{1}{C}\left(1-\frac{M}{L}\right)$$

þá fæst

$$m\ddot{I}_1 + (k+k_{12})I_1 - k_{12}I_2 = 0$$
$$m\ddot{I}_2 + (k+k_{12})I_2 - k_{12}I_1 = 0$$

og því

$$\omega_1 = \sqrt{\frac{k+2k_{12}}{m}} = \sqrt{\frac{1+\frac{M}{L}}{C(L-\frac{M^2}{L})}} = \sqrt{\frac{1}{C(L-M)}}$$
$$\omega_2 = \sqrt{\frac{k}{m}} = \sqrt{\frac{1-\frac{M}{L}}{C(L-\frac{M^2}{L})}} = \sqrt{\frac{1}{C(L+M)}}$$

(14)

12-21 þrír sveiflar tengdir þ.a.

$$U = \frac{1}{2}\left\{K_1(x_1^2+x_3^2) + K_2x_2^2 + K_3(x_1x_2+x_2x_3)\right\}$$

$$K_3 = \sqrt{2K_1K_2}$$

$$M = \frac{1}{2}\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \qquad A = \frac{1}{2}\begin{pmatrix} K_1 & \frac{K_3}{2} & 0 \\ \frac{K_3}{2} & K_2 & \frac{K_3}{2} \\ 0 & \frac{K_3}{2} & K_1 \end{pmatrix}$$

$$\omega_1^2 = -\frac{\sqrt{2(2K_1K_2)+K_2^2-2K_1K_2+K_1^2} - K_1 - K_2}{2m} = 0$$

$$\omega_2^2 = \frac{\sqrt{2(2K_1K_2)+K_2^2-2K_1K_2+K_1^2} + K_2 + K_1}{2m} = \frac{K_2+K_1}{m}$$

$$\omega_3^2 = \frac{K_1}{m}$$

(15)

Við fáum því eigintíðnirnar og normalsveifluháttina

$$\omega_1 = 0 \longleftrightarrow \bar{\eta}_1 \sim \left(1, -\tfrac{1}{2}\sqrt{\tfrac{K_1}{K_2}}, 1\right)$$
$$\omega_2 = \sqrt{\frac{K_2+K_1}{m}} \longleftrightarrow \bar{\eta}_2 \sim \left(1, +\tfrac{1}{2}\sqrt{\tfrac{K_2}{K_1}}, 1\right)$$
$$\omega_3 = \sqrt{\frac{K_1}{m}} \longleftrightarrow \bar{\eta}_3 \sim (1, 0, -1)$$

$\omega_1 \to \ddot{\bar{\eta}}_1 = 0$ með lausn $\bar{\eta}_1(t) = at + b$

því $\ddot{\bar{\eta}}_i + \omega_i \bar{\eta} = 0$ eða öfugt

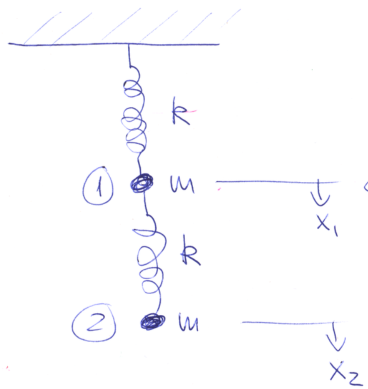
þessu fylgir líka $\bar{\nabla} U = (K_1x_1 + K_3x_3, K_2x_2 + K_3(x_1+x_3), K_1x_3 + K_3x_2)$

og $\bar{\nabla} U\Big|_{(x,y,z)=\alpha\cdot\bar{\eta}_1} = (0,0,0)$ fyrir öll $\alpha \in \mathbb{R}$

leiðir til sömu ákvörðu sem fæst til að finna eigingildin

(16)

12-07



þyngdarkraftarnir er inni í jafnvægis-stöðu massanna

$$m\ddot{x}_1 = -kx_1 + k(x_2-x_1) = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = -k(x_2-x_1) = -kx_2 + kx_1$$

Reynum lausn $\quad x_1(t) = Ae^{i\omega t}, \quad x_2(t) = Be^{i\omega t}$

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$$\rightarrow \quad \begin{aligned} -m\omega^2 A + 2kA - kB &= 0 \\ -m\omega^2 B + kB - kA &= 0 \end{aligned}$$

$$\begin{pmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$\rightarrow \quad (-m\omega^2 + 2k)(-m\omega^2 + k) - k^2 = 0$$

$$m^2\omega^4 - 2km\omega^2 - km\omega^2 + 2k^2 - k^2 = 0$$

$$m^2\omega^4 - 3km\omega^2 + k^2 = 0$$

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$$\omega^4 - \frac{3k}{m}\omega^2 + \left(\frac{k}{m}\right)^2 = 0$$

$$\omega_{12}^2 = \left\{ \frac{3}{2} \pm \frac{\sqrt{5}}{2} \right\} \frac{k}{m}$$

$$\omega_1 = \sqrt{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)\frac{k}{m}}, \quad \omega_2 = \sqrt{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)\frac{k}{m}}$$

$$a_1 \sim \left(1, \frac{1+\sqrt{5}}{2}\right)$$

$$a_2 \sim \left(1, \frac{1-\sqrt{5}}{2}\right)$$

Ef x_2 er fast, $x_2 = 0$

$$\rightarrow \quad m\ddot{x}_1 = -2kx_1$$

$$\rightarrow \quad \omega_1^0 = \sqrt{\frac{2k}{m}}$$

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Ef x_1 er fast, $x_1 = 0$

$$\rightarrow \quad m\ddot{x}_2 = -kx_2$$

$$\rightarrow \quad \omega_2^0 = \sqrt{\frac{k}{m}}$$

Berum saman

gullinsnið....

$$\omega_1 = \sqrt{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)\frac{k}{m}} \simeq 0{,}618 \cdot \sqrt{\frac{k}{m}} \qquad \omega_1^0 \simeq 1{,}414 \cdot \sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)\frac{k}{m}} \simeq 1{,}618 \cdot \sqrt{\frac{k}{m}} \qquad \omega_2^0 = \sqrt{\frac{k}{m}}$$

Tíðnir óvíxlverkandi kerfisins finnast ekki í víxlverkandi kerfinu, víxlverkun klýfur eigintíðnum

20

Eiginvigrar víxlverkandi kerfis

$a_1 \sim (1, 1{,}618)$ – samhverfur sveifluháttur

$a_2 \sim (1, -0{,}618)$ andsamhverfur sveifluháttur

hærri orka vegna → hreyfingar ~~mið~~ í gormi