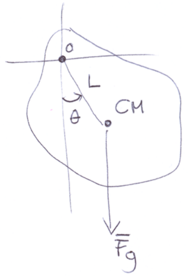
11-32 leysa Ex. 11.2 þegar sveiflur eru ekki smáar (1)



$\theta(0) = 67°$

finna $\dot{\theta}(t)$ þegar $\theta = 1°$

$M = 340\,g$, $L = 13\,cm$, $k = 17\,cm$

$T = \frac{I}{2}\dot{\theta}^2$, $U = -MgL\cos\theta$

$I = Mk^2$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \rightarrow -MgL\sin\theta - I\ddot{\theta} = 0$$

$$\rightarrow \ddot{\theta} + \frac{MgL}{Mk^2}\sin\theta = 0$$

eða

$$\boxed{\ddot{\theta} + \frac{gL}{k^2}\sin\theta = 0}$$

(2)

Umskrifum hreyfijöfnu

$$\frac{d}{dt}(\dot{\theta}) = \frac{gL}{k^2}\frac{d(\cos\theta)}{d\theta} \rightarrow d\theta\,\frac{d}{dt}(\dot{\theta}) = \frac{gL}{k^2}d(\cos\theta)$$

$$\rightarrow \dot{\theta}\,d\dot{\theta} = \frac{gL}{k^2}d(\cos\theta)$$

Heildum

$$\int_{\dot{\theta}(0)=0}^{\dot{\theta}} \dot{\theta}'\,d\dot{\theta}' = \frac{gL}{k^2}\int_{\cos\theta_0}^{\cos\theta} d(\cos\theta)' \rightarrow \frac{\dot{\theta}^2}{2} = \frac{gL}{k^2}\left\{\cos\theta - \cos\theta_0\right\}$$

$$\rightarrow \boxed{\dot{\theta} = \sqrt{\frac{2gL}{k^2}(\cos\theta - \cos\theta_0)}}$$

Höfum séð áður að $\dot{\theta}$ og því líka lotan eru háð útslaginu

$\dot{\theta} = \dot{\theta}(\theta, \theta_0)$, $\dot{\theta}(1°, 67°) = \sqrt{\frac{2 \cdot 9.81 \cdot 0.13}{(0.17)^2}(\cos 1° - \cos 67°)} \simeq 7{,}33$

11-27 x_3 er samhverfuás hlutar, engir kraftar ~~eða~~ vogi (3)

L hverfiþungi um x_3'

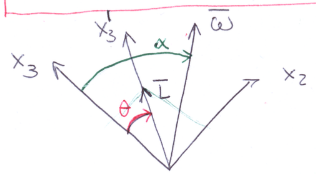
$I_1 = I_2$

Hornið milli $\bar{\omega}$ og x_3 er α

$\bar{\omega}$ og L eru upphaflega í x_2-x_3-sléttu

$\boxed{\text{Hver er hornhraði } x_3 \text{ um } L \text{ (m.v. } I_1, I_3, \omega, \alpha)}$?

$\rightarrow \dot{\phi}$?



$$L_1 = I_1\omega_1 = 0$$

$$L_2 = I_2\omega_2 = I_1\omega_2 = L\sin\theta = I_1\omega\sin\alpha$$

$$L_3 = I_3\omega_3 = L\cos\theta = I_3\omega\cos\alpha$$

(4)

Lausn vogislausa snúðsins segir að L sé í sléttu $\bar{\omega}$ og $\hat{e}_3$ $\rightarrow$ $\hat{e}_3$ og $\bar{\omega}$ velta um $L \parallel \hat{e}_3'$

þegar $\hat{e}_2$ er í sléttu $\hat{e}_3, \bar{\omega}$ og L ← upphafsgildi

$\rightarrow \psi = 0$ og (11.102) gefur $\omega_2 = \dot{\phi}\sin\theta$

$$\rightarrow \dot{\phi} = \frac{\omega_2}{\sin\theta} = \frac{\omega\sin\alpha}{\sin\theta}$$ (sem er líka jafna (11.147) í bók)

En við höfum frá síðu (3)

$$\tan\theta = \frac{L_2}{L_3} = \frac{I_1}{I_3}\tan\alpha$$

$$\omega_3 = \omega\cos\alpha$$

Og hornafræðin

$$\sin\theta = \pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\rightarrow \dot{\phi} = \frac{\omega\sin\alpha}{\sin\theta} = \pm\,\omega\sin\alpha\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$$

(5)

$$\dot{\phi} = \pm \omega \sin\alpha \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$$

$$= \pm \omega \sin\alpha \frac{\sqrt{1+\left(\frac{I_1}{I_3}\right)^2\tan^2\alpha}}{\frac{I_1}{I_3}\tan\alpha}$$

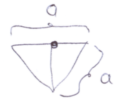
$$= \pm \omega \sin\alpha \frac{I_3}{I_1}\sqrt{\cot^2\alpha + \left(\frac{I_1}{I_3}\right)^2}$$

$$= \pm \omega \frac{I_3}{I_1}\sqrt{\cos^2\alpha + \left(\frac{I_1}{I_3}\right)^2\sin^2\alpha}$$

$$= \pm \frac{\omega}{I_1}\sqrt{I_3^2\cos^2\alpha + I_1^2\sin^2\alpha}$$

(6)

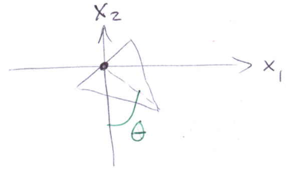
11-24 Jafn arma þríhyrningur hengdur upp:

a)  b) 

Hliðarlengd a

$\hookrightarrow$ hæð h: $\sqrt{\frac{a^2}{4}+h^2} = a \to h = \sqrt{a^2-\frac{a^2}{4}} = \frac{\sqrt{3}a}{2}$

flötur: $A = \frac{a}{2}\cdot h = \frac{\sqrt{3}a^2}{4} \to \rho = \frac{m}{A} = \frac{4m}{\sqrt{3}a^2}$

a) 

$$I_3 = \rho\int dxdy\,(x^2+y^2)$$

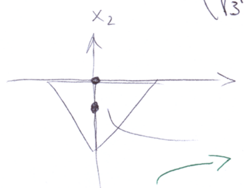
$$= 2\rho\int_0^{a/2}dx\int_{-\frac{\sqrt{3}}{2}(a-2x)}^{0}dy\,(x^2+y^2)$$

(7)

$$I_3 = 2\rho\int_0^{a/2}dx\left\{\left(x^2y+\frac{y^3}{3}\right)\Big|_{-\frac{\sqrt{3}}{2}(a-2x)}^{0}\right\}$$

$$= 2\rho\int_0^{a/2}dx\left\{x^2\cdot\frac{\sqrt{3}}{2}(a-2x)+\frac{1}{3}\left(\frac{\sqrt{3}}{2}(a-2x)\right)^3\right\} = 2\rho\frac{a^4}{16\sqrt{3}}$$

$$= 2\left(\frac{4m}{\sqrt{3}a^2}\right)\frac{a^4}{16\sqrt{3}} = \frac{m}{6}a^2$$



þegar $\theta = 0$

$$y_{cm} = \frac{2\rho}{m}\int_0^{a/2}dx\int_{-\frac{\sqrt{3}}{2}(a-2x)}^{0}y\,dy$$

$$= -\frac{2\rho}{m}\int_0^{a/2}dx\left\{\frac{1}{2}\left(\frac{\sqrt{3}}{2}(a-2x)\right)^2\right\} = -\frac{\rho}{m}\frac{a^3}{8}$$

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$$y_{cm} = -\frac{\rho}{m}\frac{a^3}{8} = -\frac{4}{\sqrt{3}a^2}\frac{a^3}{8} = -\frac{a}{2\sqrt{3}}$$

Hreyfiorka: $T = \frac{I_3}{2}\dot{\theta}^2 = \frac{m}{12}a^2\dot{\theta}^2$ lengd sveifluáss

Stöðuorka: $U = mg|y_{cm}|(1-\cos\theta) = +\frac{mga}{2\sqrt{3}}(1-\cos\theta)$

$$\to L = T - U = \frac{m}{12}(a\dot{\theta})^2 - \frac{mga}{2\sqrt{3}}(1-\cos\theta)$$

því leiðir Euler-Lagrange til hreyfijöfnu

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = 0$$

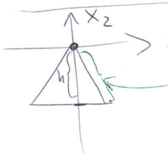
$$\to -\frac{mga}{2\sqrt{3}}\sin\theta - \frac{ma^2}{6}\ddot{\theta} = 0$$

$\rightarrow$ $\boxed{\ddot{\theta} + \sqrt{3}\,\frac{g}{a}\sin\theta = 0}$

berum saman við einfaldan pendúl

$\ddot{\theta} + \omega^2 \sin\theta = 0$

$\rightarrow$ $\omega = \sqrt{\sqrt{3}\,\frac{g}{a}}$



$$I'_3 = 2\rho \int_{-\frac{\sqrt{3}a}{2}}^{0} dy \int_{-\frac{y}{\sqrt{3}}}^{0} dx\,(x^2+y^2)$$

$-h$

$$= 2\rho \int_{-\frac{\sqrt{3}a}{2}}^{0} dy \left\{ \left(\frac{x^3}{3} + xy^2\right)\Big|_0^{-y/\sqrt{3}} \right\} = -2\rho \int_{-\frac{\sqrt{3}a}{2}}^{0} dy \left\{ \frac{y^3}{3(\sqrt{3})^3} + \frac{y^3}{\sqrt{3}} \right\}$$

$$I'_3 = 2\rho \left\{ \frac{\left(\frac{\sqrt{3}a}{2}\right)^4}{4\cdot 3(\sqrt{3})^3} + \frac{\left(\frac{\sqrt{3}a}{2}\right)^4}{4\cdot\sqrt{3}} \right\} = 2\rho a^4 \left\{ \frac{1}{12\cdot 16} + \frac{3}{4\cdot 16} \right\}\sqrt{3}$$

$$= 2\rho a^2 \sqrt{3}\,\frac{10}{12\cdot 16} = a^2 m\,\frac{10}{6\cdot 4} = \frac{5}{12} m a^2$$

og því

$$L = \frac{5}{24} m (a\dot{\theta})^2 - U$$

þar sem

áður

$$U = mg(1-\cos\theta)\cdot(h - Y_{cm}) = mg\,\frac{a}{\sqrt{3}}(1-\cos\theta)$$

$$\rightarrow L = \frac{5}{24} m (a\dot{\theta})^2 - mg\,\frac{a}{\sqrt{3}}(1-\cos\theta)$$

$$\rightarrow \ddot{\theta} + \frac{12g}{5\sqrt{3}a}\sin\theta = 0 \quad \rightarrow \omega = \sqrt{\frac{12}{5\sqrt{3}}\frac{g}{a}}$$

11-21 (11.54) – (11.61) í fylgja framsetningu

hvernig var þetta athugað í fyrirlestri 18

Endurbætum $\bar{L} = \mathbb{I}\bar{\omega}$

fylkið sem stendur fyrir þinum

vigrar

$\bar{I}$ snúnu hnitakerfi

$\bar{L}' = \mathbb{I}'\bar{\omega}'$

$\bar{x}' = \lambda\bar{x}$

$\bar{x} = \lambda^t\bar{x}'$

$\lambda\lambda^t = 1$

$\lambda^t = \lambda^{-1}$

hornrétt fylki

$\bar{L} = \lambda^t\bar{L}'$, $\bar{\omega} = \lambda^t\bar{\omega}'$

$\lambda^t\bar{L}' = \lambda^t\mathbb{I}'\lambda\bar{\omega}$

$\bar{L} = (\lambda^t\mathbb{I}'\lambda)\bar{\omega}$ $\rightarrow$ $\mathbb{I} = \lambda^t\mathbb{I}'\lambda$

og

$\lambda\mathbb{I}\lambda^t = \mathbb{I}'$

11-29 Samhverfur snúður þegar x'_3 og x_3 eru í sömu átt

Sjá mynd 11-15 $\rightarrow$ $\theta = 0$

$$\left.\begin{aligned} \rightarrow p_\phi &= I_3\{\dot{\phi} + \dot{\psi}\} \\ p_\psi &= I_3\{\dot{\phi} + \dot{\psi}\} \end{aligned}\right\} = I_3\omega_3$$

og (11.159) $\rightarrow$

fyrir ákveðnar fast $(\theta = 0, \dot{\theta} = 0)$

$$E = \frac{I_3}{2}\omega_3^2 + Mgh$$

$$\rightarrow E' = E - \frac{I_3}{2}\omega_3^2 = Mgh$$

Hvað með stöðugleika snúðsins?

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Notum (11.161)

$$E' = \frac{I_1}{2}\dot\theta^2 + \frac{\{P_\phi - P_\psi\cos\theta\}^2}{2I_1\sin^2\theta} + Mgh\cos\theta = Mgh$$

$$= \boxed{\frac{I_1}{2}\dot\theta^2 + \frac{(I_3\omega_3)^2\{1-\cos\theta\}^2}{2I_1\sin^2\theta} + Mgh\cos = Mgh}$$

Breytuskipti $z = \cos\theta$

$$\frac{d\cos\theta}{dt} = -\sin\theta\frac{d\theta}{dt} \quad \to \quad \frac{dz}{dt} = -\sin\theta\frac{d\theta}{dt}$$

$$\frac{I_1}{2}\frac{\dot z^2}{\sin^2\theta} + \frac{(I_3\omega_3)^2(1-z)^2}{2I_1\sin^2\theta} - Mgh\{1-z\} = 0$$

(14)

$$\frac{I_1}{2}\frac{\dot z^2}{(1-z^2)} + \frac{(I_3\omega_3)^2(1-z)^2}{2I_1(1-z^2)} - Mgh(1-z) = 0$$

$$\to \dot z^2 = \frac{(I_3\omega_3)^2(1-z)^2}{I_1^2} - \frac{2Mgh(1-z)(1-z^2)}{I_1}$$

$$= \frac{(1-z)^2}{I_1^2}\left\{2MghI_1(1+z) - (I_3\omega_3)^2\right\}$$

þurfum $\dot z^2 \geq 0$, gerum ráð fyrir að snúðurinn snúist hratt þ.a. $\{\ldots\} < 0 \to z = 1 \to \theta = 0$

$\to$ stöðug hreyfing ef $\to 4MghI_1 - (I_3\omega_3)^2 < 0$

$$\to \boxed{\frac{4MghI_1}{(I_3\omega_3)^2} < 1}$$

Síðan má stöðuleika gerast ef ω_3 er ekki stórt ...

(15)

(11-31) þunn einsleit plata x - kerfi plötu

I_1, $I_2 > I_1$ um höfuðásana

$I_3 = I_1 + I_2$

O og O' falla saman við CM plötu

$t = 0$ sett á snúning í kraftfríu umhverfi með Ω um ás sem hallar α frá flöt sléttu og þvert á x_2-ás

Ef $\frac{I_1}{I_2} = \cos(2\alpha)$ sýna að

$\omega_2(t) = \Omega\cos\alpha \cdot \tanh\{\Omega t\sin\alpha\}$

(16)

$I_1 = I_2\cos(2\alpha)$

$I_3 = I_1 + I_2 = I_2\{1+\cos(2\alpha)\} = 2I_2\cos^2\alpha$

$I_1 - I_2 = I_2\{\cos(2\alpha) - 1\} = -I_2\{1-\cos(2\alpha)\}$

$= -2I_2\sin^2\alpha$

Kraftfrítt umhverfi

(11.114)

$$\{I_2 - I_3\}\omega_2\omega_3 - I_1\dot\omega_1 = 0 \quad (1)$$

$$\{I_3 - I_1\}\omega_3\omega_1 - I_2\dot\omega_2 = 0 \quad (2)$$

$$\{I_1 - I_2\}\omega_1\omega_2 - I_3\dot\omega_3 = 0 \quad (3)$$

Nýtum upplýsingarnar um II

$$\{-2I_2\sin^2\alpha\}\omega_1\omega_2 - 2I_2\cos^2\alpha\cdot\dot\omega_3 = 0 \quad (3)$$

$$I_2\{1-2\cos^2\alpha\}\omega_2\omega_3 - I_2\cos(2\alpha)\cdot\dot\omega_1 = 0 \quad (1)$$

$$I_2\{2\cos^2\alpha-\cos(2\alpha)\}\omega_3\omega_1 - I_2\dot\omega_2 = 0 \quad (2)$$

$(3) \rightarrow \dot\omega_3 = -\omega_1\omega_2\tan^2\alpha$

$(1) \rightarrow \dot\omega_1 = -\omega_2\omega_3$

$(2) \rightarrow \dot\omega_2 = \omega_3\omega_1$

$$\rightarrow \omega_1\omega_2\omega_3 = \dot\omega_2\omega_2 \quad \leftarrow (2)$$

$$= -\dot\omega_1\omega_1 \quad \leftarrow (1)$$

$$= -\dot\omega_3\cdot\omega_3\cot^2\alpha \quad \leftarrow (3)$$

Hvaðan

$$\omega_2^2-\omega_2^2(0) = -\omega_1^2+\omega_1^2(0) = -\omega_3^2\cot^2\alpha+\omega_3^2(0)\cdot\cot^2\alpha$$

upphafsgildi

$\omega_2(0) = 0$ ← punkt á x_2

$\omega_1(0) = \Omega\cos\alpha$

$\omega_3(0) = \Omega\sin\alpha$

Nú fæst

$$\omega_2^2 = -\omega_1^2+\Omega^2\cos^2\alpha = -\omega_3^2\cot^2\alpha+\Omega^2\cos^2\alpha$$

$(2) \rightarrow \dot\omega_2^2 = \omega_3^2\omega_1^2$ og $\omega_1^2 = \omega_3^2\cot^2\alpha$

$$\dot\omega_2 = \omega_3^2\cot\alpha$$

$$\omega_3^2 = \Omega^2\sin^2\alpha-\omega_2^2\tan^2\alpha$$

$$\dot\omega_2 = -(\cot\alpha)\cdot\{\omega_2^2\tan^2\alpha-\Omega^2\sin^2\alpha\}$$

$$\rightarrow \frac{\dot\omega_2}{\{\omega_2^2\tan^2\alpha-\Omega^2\sin^2\alpha\}} = -\cot\alpha$$

Nýtum at $\dot\omega_2 = \frac{d\omega_2}{dt}$

$$\rightarrow \int_0^{\omega_2}\frac{d\omega_2'}{\omega_2'^2\tan^2\alpha-\Omega^2\sin^2\alpha} = \cot\alpha\int_0^t dt$$

ef $\omega_2^2\tan^2\alpha < \Omega^2\sin^2\alpha$

$$(E.4c) \rightarrow -\frac{1}{\tan\alpha\cdot\Omega\sin\alpha}\mathrm{Artanh}\left\{\frac{\omega_2\tan\alpha}{\Omega\sin\alpha}\right\} = -t\cos\alpha$$

$$\rightarrow \boxed{\omega_2(t) = \Omega\cos\alpha\cdot\tanh\{\Omega t\sin\alpha\}}$$

sum er rætt