(1)

11-12 Sýna að engin hverfitregða um höfuðás geti verið meiri en summa hinna tveggja

Hugsum um I_i

$$I_j + I_k = \int dv\{x_i^2 + x_k^2\}\rho + \int dv\{x_i^2 + x_j^2\}\rho$$

þar sem við höfum notað (11.15)

$$I_{ij} = \int_V dv\, \rho(r)\left\{\delta_{ij}\sum_k x_k^2 - x_i x_j\right\}$$

og $I_i = I_{ii}, \ldots$

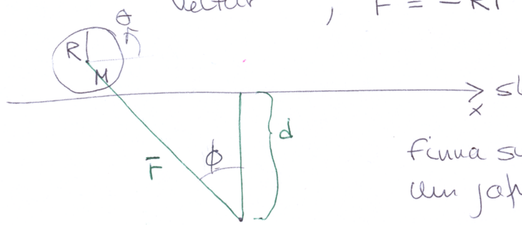
$$I_j + I_k = \int dv\{x_j^2 + x_k^2\}\rho + 2\int dv\, x_i^2 \rho$$

$$= I_i + 2\underbrace{\int dv\, x_i^2 \rho}_{>0}$$

(2)

því fast að $I_i \le I_j + I_k$

11-07 Veltur, $\bar F = -k\bar r$



Finna sveiflutíma dóts um jafnvægispunktinn

Velta eftir x-ás → kraftarinn á CM stefnir er

$F_x = -kr\sin\phi = -kx$, $I = \frac{MR^2}{2}$, $\to U = \frac{kx^2}{2}$

Hreyfiorkan er

$$T = \frac{M}{2}\dot x^2 + \frac{I}{2}\dot\theta^2 = \frac{M}{2}\dot x^2 + \frac{M(R\dot\theta)^2}{4}$$

En $R\theta = x \to T = \frac{3M}{4}\dot x^2$

(3)

$$L = T - U = \frac{3M}{4}\dot x^2 - \frac{kx^2}{2}$$

Notum jöfnu Lagrange

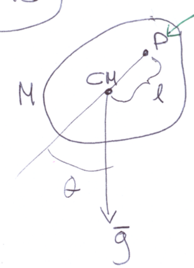
$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) = 0 \to -kx - \frac{d}{dt}\left(\frac{3M}{2}\dot x\right) = 0$$

$$\to \frac{3}{2}M\ddot x + kx = 0 \quad \text{eða} \quad \ddot x + \frac{2k}{3M}x = 0$$

og því $\omega = \sqrt{\frac{2k}{3M}}$

(4)

11-15



upphengipunktur

Setning Steiners (11.49)

$\to I = MR_0^2 + Ml^2$

þekkjum ekki endilega R_0

$$T = \frac{I}{2}\dot\theta^2, \quad U = Mgl(1-\cos\theta)$$

$$\to L = \frac{I}{2}\dot\theta^2 - Mgl(1-\cos\theta)$$

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = 0 \to -Mgl\sin\theta - \frac{I}{2}2\ddot\theta = 0$$

< lítil sveifla

$$\to \ddot\theta + \underbrace{\frac{Mgl}{I}}\theta = 0$$

$$= \frac{Mgl}{MR_0^2 + Ml^2} = \frac{gl}{R_0^2 + l^2} = \omega^2$$

Ef við finnum annan punkt fyrir upphengju með sama ω með l' þá gildir

$$\frac{gl}{R_0^2+l^2} = \frac{gl'}{R_0^2+(l')^2}$$

$$\rightarrow \quad l\{R_0^2+(l')^2\} = l'\{R_0^2+l^2\}$$

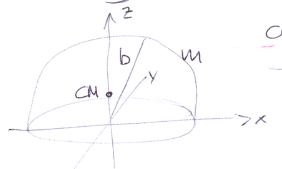
$$\rightarrow \quad R_0^2\{l-l'\} = l^2 l' - (l')^2 l = ll'\{l-l'\} \quad \rightarrow \quad \underline{R_0^2 = ll'}$$

$$\rightarrow \quad \omega = \sqrt{\frac{gl}{R_0^2+l^2}} = \sqrt{\frac{gl}{ll'+l^2}} = \sqrt{\frac{g}{l'+l}}$$

Þarf ekki að mæla I, finna þarf tvo punkta með sömu tíðni

11-14

Gegnheilt hálfhvel, finna hverfitregðu höfuðása, og höfuð ása m.v. CM



Kartísk hnit í gegnum miðju kúlunnar eru (x,y,z). Miðað við þau er hverfitregðufylkið $\mathbb{J}$

fyrir CM vitum við að $x_{cm}=0$, $y_{cm}=0$

$$\rho = \frac{m}{V} = \frac{m}{\left(\frac{4}{3}\pi b^3\right)\frac{1}{2}} = \frac{3m}{2\pi b^3} \quad \text{fasti}$$

$r\cos\theta = z$

$$Z_{cm} = \frac{1}{m}\int dv\,\rho\cdot z = \frac{\rho}{m}2\pi\int_0^{\pi/2}\sin\theta d\theta\cdot\cos\theta\int_0^b r^2dr\,r$$

$$= \frac{3}{2\pi b^3}2\pi\left(\frac{1}{2}\right)\frac{b^4}{4} = \frac{3b}{8}$$

Samhverfa $\mathbb{J}_{11} = \mathbb{J}_{22}$

$$\mathbb{J}_{11} = \rho\int dv\{y^2+z^2\}$$

$$= \rho\int_0^b r^2dr\int_0^{\pi/2}d\theta\cdot\sin\theta\int_0^{2\pi}d\phi\,\{\sin^2\theta\sin^2\phi+\cos^2\theta\}r^2$$

$$= \frac{3m}{2\pi b^3}\frac{b^5}{5}\int_0^{\pi/2}d\theta\sin\theta\,\{\pi\sin^2\theta+2\pi\cos^2\theta\}$$

$$= \frac{3m}{2\pi b^3}\frac{b^5}{5}\left[\pi\cdot\frac{2}{3}+2\pi\cdot\frac{1}{3}\right] = \frac{mb^2}{2\cdot 5}\{2+2\}$$

$$= \frac{2mb^2}{5}$$ sem er líka $\mathbb{J}_{22}$

$$\mathbb{J}_{33} = \rho\int dv\{x^2+y^2\} = \rho\int_0^b r^4dr\int_0^{2\pi}d\phi\int_0^{\pi/2}\sin\theta d\theta$$

$$\cdot\left\{\sin^2\theta\underbrace{\left(\sin^2\phi+\cos^2\phi\right)}_{=1}\right\}$$

$$= \frac{3m}{2\pi b^3}\frac{b^5}{5}2\pi\int_0^{\pi/2}d\theta\,\sin^3\theta = \frac{3m}{2\pi b^3}\frac{b^5}{5}2\pi\cdot\frac{2}{3}$$

$$= \frac{2mb^2}{5}$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$\rightarrow$ $\mathbb{J}_{ij} = 0$ ef $i\neq j$

$\mathbb{J}$ er á hornalínuham

Þarf um $\mathbb{I}$ ~~miðað~~ við CM

$$\mathbb{I}_{33} = \mathbb{J}_{33}$$

en fyrir hin er notuð setning Steiners

$$\mathbb{I}_{11} = \mathbb{J}_{11} - m z_{CM}^2 = \mathbb{J}_{11} - m\left(\frac{3b}{8}\right)^2$$

$$= \frac{2}{5} mb^2 - m\frac{9b^2}{64} = \frac{83}{320} mb^2 = \mathbb{I}_{22}$$

og áður var komið

$$\mathbb{I}_{33} = \frac{2mb^2}{5}$$

(x',y',z')

Höfuðásarnir eru því hnitakerfið, Kartíska, sem lyftir upp í z_{CM}, en ásarnir eru samsíða (x,y,z)

11-11

a) velta án þess að hann renni til

finna hornhraða þegar kubbur „lendir"

① Hæð CM yfir sléttu $\frac{l}{\sqrt{2}}$

② — || — $\frac{l}{2}$

① Einungis stöðuorka $U_1 = mg\frac{l}{\sqrt{2}}$

② stöðuorka $U_2 = mg\frac{l}{2}$

Hreyfiorka $T_2 = \frac{m}{2}v_{CM}^2 + \frac{I}{2}\omega^2$

Af mynd sést að $v_{CM} = \frac{l}{\sqrt{2}}\omega$

samkvæmt Ex. 11.5 í bók er I hér um CM þvert á eina flötinn $I = \frac{m}{6}l^2$, og Steiner leiðir þá til I' um ásinn í gegnum P

því fast frá orku varðveislu

$$I' = I + m\left(\frac{l}{\sqrt{2}}\right)^2 = \frac{2}{3}ml^2$$

eða $U_1 = U_2 + T_2$

Sleppum hliðrun massamiðju, einungis snúningur um P

$$mg\frac{l}{\sqrt{2}} = mg\frac{l}{2} + 0 + \frac{2m}{6}l^2\omega^2$$

$$\to gl\left\{\frac{1}{\sqrt{2}} - \frac{1}{2}\right\} = (\omega l)^2\left\{0 + \frac{1}{3}\right\} = (\omega l)^2\frac{1}{3}$$

$$gl\left\{\sqrt{2} - 1\right\} = (\omega l)^2\frac{2}{3} \to \omega^2 = \frac{3g}{2l}\left\{\sqrt{2} - 1\right\}$$

b) Enginn núningskraftur

Engir láréttir kraftar
CM fellur lóðrétt
í fallinu sjálfu

Kubburinn um θ frá 0 í $\frac{\pi}{4}$

$\to y_{CM} = \frac{l}{\sqrt{2}}\cos\theta$ (Hreyfingin er ekki með jöfnum hornhraða?)

$\dot{y}_{CM} = -\frac{l}{\sqrt{2}}\dot{\theta}\sin\theta$

og í lokapunktinum gildir $(\dot{y}_{CM})_2 = -\frac{1}{2}l\dot{\theta} = -\frac{l}{2}\omega$

þegar $\theta = \pi/4$

(13)

Orkuvarðveislan er þá

$$U_1 = U_2 + T_2$$

$$mg\frac{l}{\sqrt{2}} = \frac{mgl}{2} + \frac{m}{2}\left(\dot{y}\right)_2^2 + \frac{I}{2}\omega^2$$

miða I um miðjuna

$$= \frac{mgl}{2} + \frac{m}{2}\left(-\frac{l\omega}{2}\right)^2 + \frac{ml^2}{12}\omega^2$$

$$\rightarrow \quad gl\left\{\frac{1}{\sqrt{2}} - \frac{1}{2}\right\} = (\omega l)^2\left\{\frac{1}{8} + \frac{1}{12}\right\}$$

eða

$$gl\left\{\sqrt{2} - 1\right\} = (\omega l)^2\left\{\frac{1}{4} + \frac{1}{6}\right\} = (\omega l)^2\,\frac{5}{12}$$

$$\rightarrow \quad \omega^2 = \frac{12}{5}\frac{g}{l}\left\{\sqrt{2} - 1\right\}$$

(14)

11-13

Þrír massar í hnitum

finna I, höfuðása og höfuðhverfitregður

$$m_1 = 3m \text{ í } (b,0,b)$$
$$m_2 = 4m \text{ í } (b,b,-b)$$
$$m_3 = 2m \text{ í } (-b,b,0)$$

$$I_{11} = \sum_\alpha m_\alpha\left\{x_{\alpha 2}^2 + x_{\alpha 3}^2\right\} = 3mb^2 + 4m(2b^2) + 2mb^2$$
$$= 13mb^2$$

$$I_{22} = 16mb^2, \quad I_{33} = 15mb^2$$

$$I_{12} = I_{21} = -\sum_\alpha m_\alpha x_{\alpha 1} x_{\alpha 2} = -4mb^2 - 2m(-b^2) = -2mb^2$$

$$I_{13} = I_{31} = mb^2, \quad I_{23} = I_{32} = 4mb^2$$

(15)

$$I = mb^2\begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix}$$

Eigingildi

$17 - \sqrt{7} \approx 14.354\, mb^2$

$17 + \sqrt{7} \approx 19.646\, mb^2$

$10 = 10.000\, mb^2$

höfuðhverfitregður

Eiginvigrar

$(1, 1, -1)$

$(1, -\sqrt{7}-3, -\sqrt{7}-2)$

$(1, \sqrt{7}-3, \sqrt{2}-2)$

Höfuðásar

ekki normaðir, en eru með vídd b