1

8-32 Athuga stöðugleika hringbrauta fyrir kraftinn

$$F(r) = -\frac{k}{r^2} e^{-r/a}$$

samkvæmt (8,93) verður að gilda að

$$\frac{F'(r)}{F(r)} + \frac{3}{r} > 0$$

$$\rightarrow \frac{\left[\frac{2k}{r^3} e^{-r/a} + \frac{k}{ar^2} e^{-r/a}\right]}{-\frac{k}{r^2} e^{-r/a}} + \frac{3}{r} > 0$$

$$\rightarrow \frac{2k + \frac{rk}{a}}{-kr} + \frac{3}{r} > 0$$

$$\frac{2 + \frac{r}{a}}{-r} + \frac{3}{r} > 0$$

2

$$\rightarrow \frac{3 - 2 - \frac{r}{a}}{r} > 0 \rightarrow r < a$$

08-06 Tveir massar

m_1, m_2, $M = m_1 + m_2$

x_1, x_2, r_0 í upphafi, $v_1 = 0$, $v_2 = 0$

$r = |x_2 - x_1|$, $r < r_0$ þegar $t > 0$

Heildarorkan er þá stöðuorkan í upphafi

$$E_{total} = -G \frac{m_1 m_2}{r_0}$$

Síðar í tíma:

$$E_{total} = \frac{m_1}{2} \dot{x}_1^2 + \frac{m_2}{2} \dot{x}_2^2 - G \frac{m_1 m_2}{r}$$

3

Orkan er varðveitt, engir ytri kraftar -> heildarskriðþunginn er líka varðveittur

$$\boxed{P_{total} = m_1 \dot{x}_1 + m_2 \dot{x}_2 = 0}$$ (1)

því agnirnar voru kyrrar í upphafi

Orkuvarðveislan:

$$\boxed{-G \frac{m_1 m_2}{r_0} = \frac{m_1}{2} \dot{x}_1^2 + \frac{m_2}{2} \dot{x}_2^2 - G \frac{m_1 m_2}{r}}$$ (2)

Notum (1) $\rightarrow m_1 \dot{x}_1 + m_2 \dot{x}_2 = 0 \rightarrow m_2 \dot{x}_2 = -m_1 \dot{x}_2$

í (2)

$$-G m_1 m_2 \left\{\frac{1}{r_0} - \frac{1}{r}\right\} = \frac{m_1}{2} \dot{x}_1^2 + \frac{m_1^2}{2 m_2} \dot{x}_1^2$$

$$= \frac{m_1^2}{2} \left\{\frac{\dot{x}_1^2}{m_1} + \frac{\dot{x}_1^2}{m_2}\right\} = \frac{\dot{x}_1^2}{2} m_1^2 \left\{\frac{m_2 + m_1}{m_1 m_2}\right\}$$

4

$$\dot{x}_1^2 = 2 \frac{m_1 m_2}{M} \frac{1}{m_1^2} \cdot G m_1 m_2 \left\{\frac{1}{r} - \frac{1}{r_0}\right\}$$

$$= 2 \frac{m_2^2}{M} G \left\{\frac{1}{r} - \frac{1}{r_0}\right\}$$

$$\rightarrow \dot{x}_1 = m_2 \sqrt{\frac{2G}{M} \left\{\frac{1}{r} - \frac{1}{r_0}\right\}}$$

og vegna (1) fæst því strax

$$\dot{x}_2 = -m_1 \sqrt{\frac{2G}{M} \left\{\frac{1}{r} - \frac{1}{r_0}\right\}}$$

(08-20) Sýna að eftir einni á sporbaug gildi (5)

"Ögn í þyngdarmiðju"

$$\left\langle \left(\frac{a}{r}\right)^4 \cos\theta \right\rangle = \frac{\epsilon}{(1-\epsilon^2)^{5/2}}$$

tímameðaltal

samkvæmt (8-41)

(1) $\frac{\alpha}{r} = 1 + \epsilon\cos\theta$, $\alpha = \frac{l^2}{\mu k}$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$

og (8-42) $l = \mu r^2\dot{\theta} = \text{fasti}$

(2) $a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2|E|}$ $F = -\frac{k}{r^2}$

(6)

Notum (1) og (2) til að skrifa

$$\frac{\alpha}{r} = 1 + \epsilon\cos\theta , \quad \alpha = a(1-\epsilon^2)$$

$$\rightarrow \frac{a(1-\epsilon^2)}{r} = 1 + \epsilon\cos\theta \rightarrow \frac{a}{r} = \frac{1+\epsilon\cos\theta}{(1-\epsilon^2)}$$

$$\rightarrow \left(\frac{a}{r}\right)^4 \cos\theta = \left\{\frac{1+\epsilon\cos\theta}{(1-\epsilon^2)}\right\}^4 \cos\theta$$

$$\rightarrow \left\langle \left(\frac{a}{r}\right)^4 \cos\theta \right\rangle = \frac{1}{\tau}\int_0^\tau dt \left\{\frac{1+\epsilon\cos\theta}{(1-\epsilon^2)}\right\}^4 \cos\theta$$

Eigum eftir að tengja θ og t

(7)

Annað lögmál Kepplers (8-12)

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \text{fasti}$$

og $\frac{dA}{dt} = \frac{\pi ab}{\tau}$ = $\frac{\text{flötur}}{\text{tímalotu}}$

$\alpha = a(1-\epsilon^2)$

$$dt = \frac{\tau}{\pi ab} dA = \frac{\tau}{\pi ab}\frac{r^2}{2} d\theta$$

$$= \frac{\tau}{\pi ab}\frac{\alpha^2}{2(1+\epsilon\cos\theta)^2} d\theta$$

$$\rightarrow \left\langle \left(\frac{a}{r}\right)^4 \cos\theta \right\rangle = \frac{1}{\tau}\frac{1}{(1-\epsilon^2)^2}\frac{\tau}{\pi ab}\frac{a^2}{2}\int_0^{2\pi} d\theta \cos\theta (1+\epsilon\cos\theta)^2$$

(8)

$$= \frac{1}{(1-\epsilon^2)^2}\frac{a^2}{2\pi ab}\underbrace{\int_0^{2\pi} d\theta \cos\theta (1+\epsilon\cos\theta)^2}_{2\pi\epsilon}$$

$$= \frac{1}{(1-\epsilon^2)^2}\frac{a\epsilon}{b} = \frac{\epsilon}{(1-\epsilon^2)^{5/2}}$$

$$b = \frac{\alpha}{\sqrt{1-\epsilon^2}} = \frac{a(1-\epsilon^2)}{\sqrt{1-\epsilon^2}} = a(1-\epsilon^2)^{1/2}$$

08-10

Ef braut jarðar væri hringur, hvað gerðist ef massi sólar helmingaðist allt í einu

9

Hringur $\rightarrow$ $T = \frac{m}{2}(\omega R)^2$

R: geisli brautar jarðar
m: massi jarðar
M: massi sólar

$$U = -G\frac{Mm}{R}$$

Fyrir hringbraut þarf þyngdarkraftinn að vera jafn miðsóknarkraftinum, sem er nauðsynlegur fyrir hringbraut

$$m\omega^2 R = G\frac{Mm}{R^2}$$

$$\rightarrow \quad \omega^2 = \frac{GM}{R^3}$$

10

$$\rightarrow T = \frac{m}{2}\frac{GMR^2}{R^3} = G\frac{Mm}{2R} = -\frac{1}{2}U$$

$$\rightarrow E = T + U = \frac{U}{2}$$

Ef massi sólar helmingast verður T óbreytt en U helmingast

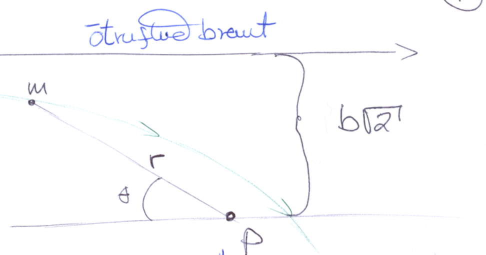
Nýja orkan verður þá

$$E_2 = T + U_2 = T + \frac{U}{2} = 0$$

$\rightarrow$ brautin breytist í fleygboga og jörðin yfirgefur sólina

08-15

11



Sýna að $r = b\,\mathrm{cotanh}\left(\frac{\theta}{\sqrt{2}}\right)$

Ef P er kraftmiðja með $F = -\frac{k}{r^5}$

Hverfiþunginn um P er fasti

Í upphafi er hann

$$l = mv(b\sqrt{2})$$

hraði massans í mikilli fjarlægð, upphafshraði

og hverfiþunginn um P er $\frac{\sqrt{km}}{b}$

$$l = \frac{\sqrt{km}}{b} = mv(b\sqrt{2})$$

hverfiþunginn er varðveittur

Til að víddin sé í lagi

12

$\rightarrow v = \sqrt{\frac{k}{2m}}\frac{1}{b^2}$, og heildarorkan sem líka er varðveitt er upphaflega hreyfiorkan

$$E = \frac{m}{2}v^2 = \frac{m}{2}\frac{k}{2}\frac{1}{b^4 m} = \frac{k}{4b^4}$$

Gefið $F(r) = -\frac{k}{r^5}$

$$\rightarrow \quad d\theta = \frac{l}{r^2}\frac{dr}{\sqrt{2m\left\{E + \frac{k}{4r^4} - \frac{l^2}{2mr^2}\right\}}}$$

$\frac{k}{4b^4}$

$U(r)$

notum $l^2 = \frac{k}{b^2}$

$$d\theta = \frac{\sqrt{km}}{br^2}\,\frac{dr}{\sqrt{\frac{km}{2b^4}+\frac{km}{2r^4}-\frac{km}{b^2r^2}}} = b\sqrt{2}\,\frac{dr}{\sqrt{r^4-2b^2r^2+b^4}}$$

$$= b\sqrt{2}\,\frac{dr}{\sqrt{(r^2-b^2)^2}} = -b\sqrt{2}\,\frac{dr}{r^2-b^2}$$

– greinin á rótinni er valin þ.s. θ vex þ. r minnkar samkvæmt myndinni á síðu 11 ......

Heildum

$$\theta = b\sqrt{2}\,\frac{1}{2b}\ln\left\{\frac{r+b}{r-b}\right\}+\theta_0 = \sqrt{2}\,\mathrm{ArCoth}\left(\frac{r}{b}\right)+\theta_0$$

þegar $r\to\infty$, sýnir myndin að $\theta_0 = 0$

skoðum $r = b\coth\left\{\frac{\theta-\theta_0}{\sqrt{2}}\right\}$ $\quad$ $r\to\infty$ þegar $\theta\to 0$ ef $\theta_0 = 0$

$$\to \quad r = b\coth\left(\frac{\theta}{\sqrt{2}}\right)$$

sjá mynd á síðu 15

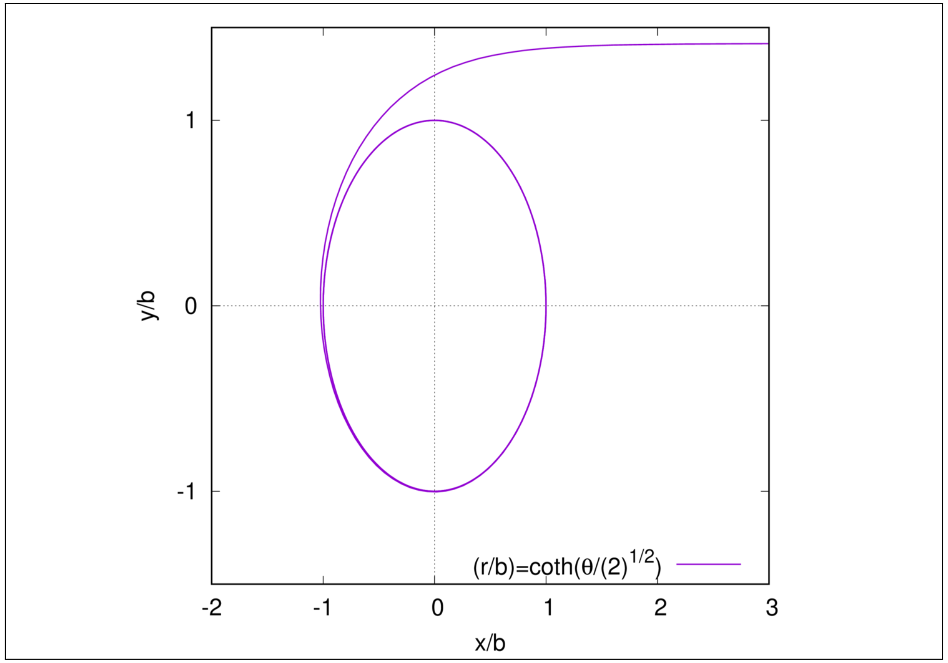
08-13 $\quad$ Skoða hreyfingu agnar í kraftsviði

$$F(r) = -\frac{k}{r^2}-\frac{\lambda}{r^3}, \quad k>0,\ \lambda>0$$

athuga tilfelli

$$\lambda < \frac{\ell^2}{\mu} \quad \text{og} \quad \lambda > \frac{\ell^2}{\mu}.$$

$$\lambda = \frac{\ell^2}{\mu}$$



Notum (8.20) þar sem $u = \frac{1}{r}$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\,\frac{1}{u^2}\,F\left(\frac{1}{u}\right)$$

sem verður þá

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2u^2}\left\{-ku^2-\lambda u^3\right\}$$

$$\to \quad \boxed{\frac{d^2u}{d\theta^2} + \left\{1-\frac{\mu\lambda}{\ell^2}\right\}u = \frac{\mu k}{\ell^2}}$$

sem er einföld línuleg afleiðujafna, en hægt er að ummynda hana í óhliðraða jöfnu

(17)

Sem sést með að skrifa

$$\frac{d^2u}{d\theta^2} + \left\{1 - \frac{\mu\lambda}{\ell^2}\right\}\left\{u - \frac{(\mu k/\ell^2)}{\left(1 - \frac{\mu\lambda}{\ell^2}\right)}\right\} = 0$$

og skilgreina nýtt fall

$$v = u - \underbrace{\frac{(\mu k/\ell^2)}{\left(1 - \frac{\mu\lambda}{\ell^2}\right)}}_{\text{bara fasti}}$$

þá fæst óhliðraða jafnan

$$\boxed{\frac{d^2v}{d\theta^2} + \left\{1 - \frac{\mu\lambda}{\ell^2}\right\} v = 0} \quad ①$$

Eins og jafnan fyrir hreintóna sveiflum en $\beta^2 = \left\{1 - \frac{\mu\lambda}{\ell^2}\right\}$ sem er sambærilegt við ω^2 í sveiflunum getur verið jákvætt, neikvætt, eða 0

(18)

$$\frac{\mu\lambda}{\ell^2} < 1 \quad \rightarrow \quad \beta^2 > 0$$

① hefur þá lausn

$$v = A\cos(\beta\theta + \delta), \quad \text{veljum } \theta_0 \text{ þ.a. } \delta = 0$$

$$v = A\cos(\beta\theta) = u - \frac{\mu k}{\ell^2 - \mu\lambda}, \quad u = \frac{1}{r}$$

$$\rightarrow \boxed{\frac{1}{r} = A\cos(\beta\theta) + \frac{\mu k}{\ell^2 - \mu\lambda}}$$

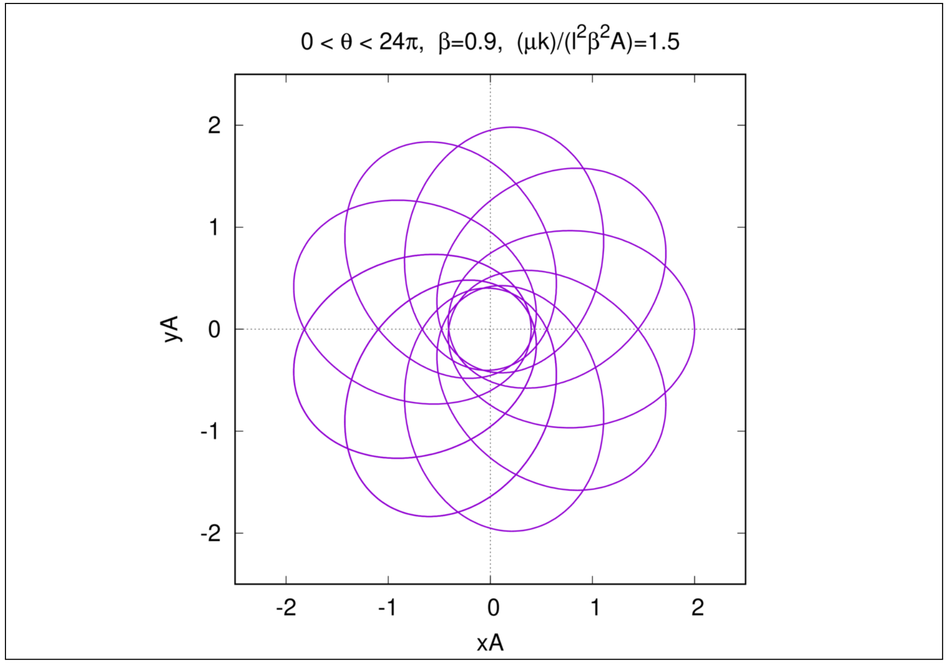
$$\frac{1}{r} = A\cos(\beta\theta) + \frac{\mu k}{\ell^2\beta^2}$$

(19)

$$\frac{1}{rA} = \cos(\beta\theta) + \frac{\mu k}{\ell^2\beta^2 A}$$

$$\rightarrow rA = \frac{1}{\cos(\beta\theta) + \frac{\mu k}{\ell^2\beta^2 A}}$$

Sjá sýningu brautarinnar á síðu (20)



(21)

$$\frac{\mu\lambda}{\ell^2} = 1 \;\longrightarrow\; \beta = 0$$

fyrir u væri jafnan þá

$$\frac{d^2u}{d\theta^2} = \frac{\mu k}{\ell^2} \;\longrightarrow\; u = \frac{\mu k}{2\ell^2}\theta^2 + A\theta + B = \frac{1}{r}$$

$\longrightarrow$ ögnin fellur inn í kraftmiðjuna með minnkandi gormhreyfingu . . . .

$$\frac{\mu\lambda}{\ell^2} > 1 \;\longrightarrow\; \beta^2 < 0$$

$$\boxed{u = A\cosh\{|\beta|\theta - \delta\}}$$

(22)

Lausnin verður

$$\frac{1}{r} = A\cosh\{|\beta|\theta\} + \frac{\mu k}{\ell^2 - \mu\lambda}$$

$\longrightarrow$ ögnin fellur í gormhreyfingu inn að kraftmiðju