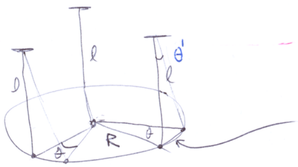
(07-20) (1)

Hringur hengdur upp í þremur punktum, jafndreifðum



snúið um lítið horn um miðju sína (vil ekki finna θ'), en þurfum strenglengd

lengd strengs er $2R\sin\left(\frac{\theta}{2}\right)$, en gjörðin hækkar þegar henni er snúið

Beinlengd niður á hana verður

$$l \to l^2 - \left(2R\sin\left(\tfrac{\theta}{2}\right)\right)^2$$

Mælum hæð gjarðar í upphafi $z = -l$, notum $z(\theta)$ og setjum $z(0) = -l$

$$\{z(\theta)\}^2 = l^2 - \left(2R\sin\left(\tfrac{\theta}{2}\right)\right)^2$$

$$\to z(\theta) = -\sqrt{l^2 - (2R)^2\sin^2\left(\tfrac{\theta}{2}\right)}$$

(2)

Fyrir lítið horn fæst þá $\sin^2\left(\frac{\theta}{2}\right) \simeq \frac{\theta^2}{4} + \cdots$

$$\to z(\theta) \simeq -l\sqrt{1 - \frac{R^2\theta^2}{l^2}} = -l\sqrt{1 - \left(\frac{R\theta}{l}\right)^2}$$

$$\simeq -l\left\{1 - \frac{1}{2}\left(\frac{R\theta}{l}\right)^2\right\}$$

þá notum við $U = mgz \simeq -mgl\left\{1 - \frac{1}{2}\left(\frac{R\theta}{l}\right)^2\right\}$

Massamiðja gjarðarinnar er kyrr, nema hvað hún getur færst til í z-átt

$$T = \frac{I}{2}\dot{\theta}^2 + \frac{m}{2}\dot{z}^2, \quad I = mR^2$$

(3)

$$\dot{z} = \frac{dz}{dt} = \frac{dz}{d\theta}\frac{d\theta}{dt} = \frac{R^2\theta}{l}\dot{\theta}$$

$$\to T = \frac{m}{2}(R\dot{\theta})^2 + \frac{m}{2}\frac{(R\theta)^2\dot{\theta}^2}{l^2}$$

$$\to L = \frac{m}{2}(R\dot{\theta})^2 + \frac{m(R\theta)^2\dot{\theta}^2}{2l^2} + mgl\left[1 - \frac{1}{2}\left(\frac{R\theta}{l}\right)^2\right]$$

þetta fall Lagrange verður aðeins fyrir hreintóna sveifil ef þessi liður ~ 0

Hlutfallið milli hans og fyrsta liðsins er $\frac{(R\theta)^2}{l^2}$ sem við leyfum okkur að jafna við núll þegar það margfaldast við $\frac{m}{2}(R\dot{\theta})^2$

$$L \simeq \frac{m}{2}(R\dot{\theta})^2 - \frac{m}{2}\frac{g}{l}(R\theta)^2 + \text{fasti}$$

(4)

Berum saman við

$$L = \frac{m}{2}(R\dot{\theta})^2 - \frac{m}{2}\omega_0^2(R\theta)^2$$

fyrir hreintóna sveiflu, þar af leiðir að hreyfijafnan verður

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

og $\omega_0 = \sqrt{\frac{g}{l}}$

07-32 (5)

$U(r) = -\frac{k}{r}$, hreyfing í mættinu í kúluhnitum

Notum jöfnu (1.100) á bls. 32 í bók

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\dot{\phi}\hat{e}_\phi$$

til að umskrifa

$$L = T - U = \frac{m}{2}v^2 + \frac{k}{r}$$

sem

$$L = \frac{m}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\cdot\dot{\phi})^2\right\} + \frac{k}{r}$$

þar sem við notum alhnitin r, θ og ϕ. Samsvarandi skriðþungar eru því

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\cdot\sin\theta$$

(6)

Fall Hamiltons er þá

$$H = p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$$

$$= \frac{m}{2}\left\{\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\cdot\dot{\phi})\right\} - \frac{k}{r}$$

$$= \left\{\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2m(r\sin\theta)^2}\right\} - \frac{k}{r}$$

Hreyfijöfnur Hamiltons

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$

$$-\dot{p}_k = \frac{\partial H}{\partial q_k}$$

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}, \quad \dot{\phi} = \frac{p_\phi}{m(r\sin\theta)^2}$$

eins og raunar var komið á síðu (5)

(7)

$$\dot{p}_r = -\frac{k}{r^2} + \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3\sin^2\theta}$$

$$\dot{p}_\theta = \frac{p_\phi^2}{mr^2(\sin^2\theta} \cdot \frac{\cos\theta}{\sin\theta)} = \frac{p_\phi^2\cot\theta}{m(r\sin\theta)^2}$$

$$\dot{p}_\phi = 0$$ ← p_ϕ er fasti, H er ekki fall af ϕ ← 4 breytur

því er í raun $H = H(r, \theta, p_r, p_\theta)$

6 fyrsta stigs hreyfijöfnur fyrir r, θ, ϕ ásamt p_r, p_θ, p_ϕ

Fjórvítt fasarúm

$$H = \left\{\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{\text{fasti}}{(r\sin\theta)^2} - \frac{k}{r}\right\}$$

Skoðum ofanvarp á r-p_r-sléttu fyrir breytilega θ

Orkuvarðveisla $H = E$

$$\rightarrow p_r = \sqrt{2mE - \frac{p_\theta^2\sin^2\theta + \text{fasti}}{(r\sin\theta)^2} + \frac{2mk}{r}}$$

07-24 (8)

Pendúll sem styttist $\frac{dl}{dt} = -\alpha$ = fasti

Veljum alhnit θ, en munum að staðsetningarhnit massa er $l\theta$

$$T = \frac{m}{2}\left(\frac{d}{dt}(l\theta)\right)^2 = \frac{m}{2}\left\{(l\dot{\theta})^2 + (\dot{l})^2\right\} = \frac{m}{2}\left\{(l\dot{\theta})^2 + (\alpha)^2\right\}$$

$$U = -mgl\cos\theta$$

$$\rightarrow L = \frac{m}{2}\left\{(l\dot{\theta})^2 + (\alpha)^2\right\} + mgl\cos\theta$$

$\vec{v} = \dot{\vec{r}} = \dot{l}\hat{e}_r + l\dot{\theta}\hat{e}_\theta$ og $\hat{e}_r \cdot \hat{e}_\theta = 0$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

þá er hægt að mynda $H = p_\theta\dot{\theta} - L$

$$\boxed{H = ml^2\dot{\theta}^2 - \frac{m}{2}\left\{(l\dot{\theta})^2 + (\alpha)^2\right\} - mgl\cos\theta = \frac{p_\theta^2}{2ml^2} - \frac{m}{2}(\alpha)^2 - mgl\cos\theta}$$

Heildarorkan er

$$E = T + U = \frac{m}{2}\left\{(l\dot\theta)^2 + (\alpha\,)^2\right\} - mgl\cos\theta$$

$$= \frac{p_\theta^2}{2ml^2} + \frac{m}{2}(\alpha\,)^2 - mgl\cos\theta$$

$$\rightarrow \quad E \neq H$$

Skoðum aðeins hreyfijöfnuna, fundna með L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}\left\{ml^2\dot\theta\right\} + mgl\sin\theta = 0$$

$$ml^2\ddot\theta + 2\dot\theta m l\dot l + mgl\sin\theta = 0$$

$$ml^2\ddot\theta - 2\dot\theta m l\alpha + mgl\sin\theta = 0$$

$$\ddot\theta - \frac{2\alpha}{l}\dot\theta + \frac{g}{l}\sin\theta = 0$$

Ef við lítum smáar sveiflur

$$\boxed{\ddot\theta - \underbrace{\frac{2\alpha}{l}\dot\theta}\, + \underbrace{\frac{g}{l}(1+0)}\,\theta = 0}$$

orku dælt inn í pendúlinn (stöðuorka hans eykst)

$$\omega_0 = \sqrt{\frac{g}{l}}$$

orkudælingin inn lætur hann sveiflast hraðar

Varúð, því jafnan leynir á sér, $l = l(t)$!

7-25 Gormferill $\quad z = R\theta,\ r = \text{fasti} = b$

Ögn í þyngdarsviði hreyfist á gormferli

Best að nota sívalningshnit r, θ, z

Almennt gildir þá $\quad T = \frac{m}{2}\left\{\dot r^2 + (r\dot\theta)^2 + \dot z^2\right\}, \quad U = mgz$

→ Með þessum brautarskorðum

$r = b$, $\dot r = 0$, höldum z, en setjum $\theta = \frac{z}{R}$

$$\rightarrow \dot\theta = \frac{\dot z}{R} \quad \text{og} \quad (r\dot\theta)^2 = b^2\frac{\dot z^2}{R^2}$$

$$\rightarrow L = \frac{m}{2}\left\{\dot z^2 + \frac{b^2}{R^2}\dot z^2\right\} - mgz$$

$$= \frac{m}{2}\left\{1 + \frac{b^2}{R^2}\right\}\dot z^2 - mgz$$

Eina alhnitið er því z

$$p_z = \frac{\partial L}{\partial \dot z} = m\left(1 + \frac{b^2}{R^2}\right)\dot z$$

$$\dot z = \frac{p_z}{m\left(1 + \frac{b^2}{R^2}\right)}$$

Fall Hamiltons verður því

$$H = p_z\dot z - L = \boxed{\frac{p_z^2}{2m\left(1-\left(\frac{b}{R}\right)^2\right)} + mgz}$$

lítur út eins og ögn í þyngdarkrafti á lóðréttri hreyfingu með virka massann $m\left(1-\left(\frac{b}{R}\right)^2\right)$

$$\dot p_z = -\frac{\partial H}{\partial z} = -mg$$

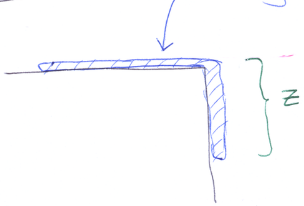
$$\dot z = \frac{\partial H}{\partial p_z} = \frac{p_z}{m\left(1+\left(\frac{b}{R}\right)^2\right)}$$

$$\rightarrow \boxed{\left(1+\left(\frac{b}{R}\right)^2\right)\ddot z = g}$$

Eins og giska mátti á þegar H eða L er skoðað

07-39 (13)

Mjög þjáll kapall, heildarlengd b



Setjum U = 0 fyrir hlutann á láréttu borðinu

$$U = g\int_z^0 dz'\,\rho(z')\,z'$$ þar sem $\rho(z) = \frac{m}{b}$ fasti

$$\rightarrow U = g\frac{m}{b}\frac{z'}{2}\Big|_z^0 = -g\frac{m}{b}\frac{z^2}{2}$$

Athugum síðan að allur kapallinn hreyfist samtímis → tregðumassinn er m, hreyfiorkan er óháð því hvort hreyfingin er lá- eða lóðrétt

$$\rightarrow T = \frac{m}{2}\dot z^2$$

(14)

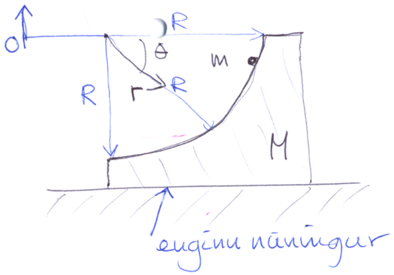
$$\rightarrow L = \frac{m}{2}\dot z^2 + \frac{gm}{2b}z^2$$

Notum Euler Lagrange

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot z}\right) = 0 \rightarrow \frac{gmz}{b} - \frac{d}{dt}(m\dot z) = 0$$

$$\rightarrow \boxed{m\ddot z - \frac{gmz}{b} = 0}$$ eða $$\boxed{\ddot z - \frac{gz}{b} = 0}$$

07-34 (15)



a) Finna hreyfijöfnur m og M

b) Finna kraft M á m

Alhnitið fyrir M
$x_M = x$ (óþarft $y_M = 0$) → $x_m = r\cos\theta + x$, $y_m = -r\sin\theta$

Nauðsynlegt ~~til~~ að nota almennt r í stað R þ.s. við viljum finna kraftinn á m í b-lið

$$T = T_M + T_m$$

$$T_M = \frac{M}{2}\dot x^2$$

$$T_m = \frac{m}{2}\{\dot x_m^2 + \dot y_m^2\} = \frac{m}{2}\{(\dot r\cos\theta - r\dot\theta\sin\theta + \dot x)^2 + (-\dot r\sin\theta - r\dot\theta\cos\theta)^2\}$$

$$U = mgy_m = -mgr\sin\theta$$

(16)

$$T_m = \frac{m}{2}\Big\{\dot r^2\cos^2\theta + \dot x^2 + r^2\dot\theta^2\sin^2\theta - \cancel{2r\dot r\dot\theta\sin\theta\cos\theta} + 2\dot r\dot x\cos\theta - 2r\dot\theta\dot x\sin\theta + \dot r^2\sin^2\theta + r^2\dot\theta^2\cos^2\theta + \cancel{2\dot r r\dot\theta\cos\theta\sin\theta}\Big\}$$

$$\rightarrow T_m = \frac{m}{2}\{\dot r^2 + (r\dot\theta)^2 + 2\dot r\dot x\cos\theta - 2r\dot\theta\dot x\sin\theta\} + \frac{m}{2}\dot x^2$$

því verður

$$\boxed{L = \frac{(M+m)}{2}\dot x^2 + \frac{m}{2}\{\dot r^2 + (r\dot\theta)^2 + 2\dot r\dot x\cos\theta - 2r\dot\theta\dot x\sin\theta\} + mgr\sin\theta}$$

Og jafnan fyrir brautarskorðunne er

$$f(x,\theta,r) = r - R = 0$$

Alhnitin eru x, θ, og r

Hreyfijöfnurnar fyrir kerfið fást með Euler-Lagrange þegar við setjum $r=R$ og $\dot r = 0$, $\ddot r = 0$ (eftir standa x og θ)

$$L' = \left(\frac{M+m}{2}\right)\dot x^2 + \frac{m}{2}\left\{(R\dot\theta)^2 - 2R\dot\theta\dot x\sin\theta\right\} + mgR\sin\theta$$

$$\frac{\partial L'}{\partial x} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot x}\right) = 0 : \quad \frac{d}{dt}\left\{(M+m)\dot x - mR\dot\theta\sin\theta\right\} = 0$$

$$\rightarrow \boxed{(M+m)\ddot x - mR\ddot\theta\sin\theta - mR\dot\theta^2\cos\theta = 0} \quad (1)$$

$$\frac{\partial L'}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot\theta}\right) = 0 : \quad -mR\dot\theta\dot x\cos\theta + mgR\cos\theta - \frac{d}{dt}\left\{mR^2\dot\theta - mR\dot x\sin\theta\right\} = 0$$

$$\rightarrow mR^2\ddot\theta - mR\ddot x\sin\theta - \cancel{mR\dot x\dot\theta\cos\theta} + \cancel{mR\dot\theta\dot x\cos\theta} - mgR\cos\theta = 0$$

$$\rightarrow \boxed{\ddot\theta - \frac{\ddot x}{R}\sin\theta - \frac{g}{R}\cos\theta = 0} \quad (2)$$

b) Brautarskorðurnar

$$f(r) = r - R = 0$$

Skorðuliður

Athugum
$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) + \lambda\frac{\partial f}{\partial r} = 0$$

$$mr\dot\theta^2 - m\dot\theta\dot x\sin\theta + mg\sin\theta - \frac{d}{dt}\left\{m\dot r + m\dot x\cos\theta\right\} + \lambda = 0$$

$$\rightarrow mr\dot\theta^2 - \cancel{m\dot\theta\dot x\sin\theta} + mg\sin\theta - m\ddot r - m\ddot x\cos\theta + \cancel{m\dot x\dot\theta\sin\theta} + \lambda = 0$$

$$mr\dot\theta^2 + mg\sin\theta - m\ddot r - m\ddot x\cos\theta + \lambda = 0$$

$$\lambda = m\ddot x\cos\theta + m\ddot r - mg\sin\theta - mr\dot\theta^2$$

Setjum núna rétta braut, $r=R$, $\dot r$, $\ddot r = 0$

$$\rightarrow \lambda = m\ddot x\cos\theta - mg\sin\theta - mR\dot\theta^2 \quad (3)$$

Notum núna (1) og (2)

$$\ddot x = \mu R\ddot\theta\sin\theta + \mu R\dot\theta^2\cos\theta, \quad \mu = \frac{m}{M+m}$$

$$\ddot\theta = \frac{\ddot x}{R}\sin\theta + \frac{g}{R}\cos\theta$$

$$\ddot x = \mu R\left\{\frac{\ddot x}{R}\sin\theta + \frac{g}{R}\cos\theta\right\}\sin\theta + \mu R\dot\theta^2\cos\theta$$
$$= \mu\ddot x\sin^2\theta + g\mu\cos\theta\sin\theta + \mu R\dot\theta^2\cos\theta$$

$$\rightarrow \ddot x(1 - \mu\sin^2\theta) = \mu R\dot\theta^2\cos\theta + g\mu\cos\theta\sin\theta$$

$$\ddot x = \frac{\mu\cos\theta\{R\dot\theta^2 + g\sin\theta\}}{1 - \mu\sin^2\theta}$$

Notum í (3)

$$\lambda = \frac{m\mu\cos^2\theta\{R\dot\theta^2 + g\sin\theta\}}{1-\mu\sin^2\theta} - mg\sin\theta - mR\dot\theta^2$$

$$= \frac{m\mu\cos^2\theta\{R\dot\theta^2 + g\sin\theta\} - mg\sin\theta + mg\mu\sin^3\theta - mR\dot\theta^2 + m\mu R\dot\theta^2\sin^2\theta}{1-\mu\sin^2\theta}$$

$$= \frac{m\mu R\dot\theta^2 - mg\sin\theta - mR\dot\theta^2 + m\mu g\cos^2\theta\sin\theta + m\mu g\sin^3\theta}{1-\mu\sin^2\theta}$$

(21)

$$\rightarrow \lambda = \frac{m\mu R\dot{\theta}^2 - mR\dot{\theta}^2 - mg\sin\theta + m\mu g\sin\theta}{1-\mu\sin^2\theta}$$

$$= \frac{m(\mu-1)\left\{g\sin\theta + R\dot{\theta}^2\right\}}{1-\mu\sin^2\theta}$$

þarsem θ_0 er upphafshorn þar sem ögnin hefur engan hraða

$\dot{\theta}$ má finna frá orkuvarðveislu

$$\left(\frac{M+m}{2}\right)\dot{x}^2 + \frac{m}{2}\left\{R^2\dot{\theta}^2 - 2\dot{x}R\dot{\theta}\sin\theta\right\} - mgR\sin\theta = -mgR\sin\theta_0$$

En hér þarf að losna við $\dot{x}$, til þess þarf að henda (1) skoðum (3)

$$\ddot{x} = \mu R\left\{\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right\} = \mu R\frac{d}{dt}\left\{\dot{\theta}\sin\theta\right\}$$

$$\frac{d}{dt}(\dot{x}) = \mu R\frac{d}{dt}\left\{\dot{\theta}\cdot\sin\theta\right\}$$

(22)

því fæst $\dot{x} = \mu R\dot{\theta}\sin\theta$, og orkuvarðveislan

$$\left(\frac{M+m}{2}\right)\mu^2R^2\dot{\theta}^2\sin^2\theta + \frac{m}{2}\left\{R^2\dot{\theta}^2 - 2\mu R^2\dot{\theta}^2\sin^2\theta\right\} - mgR\sin\theta = -mgR\sin\theta_0$$

$$\rightarrow \dot{\theta}^2 = \frac{2g(\sin\theta - \sin\theta_0)}{R(1-\mu\sin^2\theta)}, \qquad \mu = \frac{m}{M+m}$$

þetta þarf að nota í

$$\lambda = \frac{m(\mu-1)\left\{g\sin\theta + R\dot{\theta}^2\right\}}{1-\mu\sin^2\theta}$$

(23)

$$\lambda = \frac{m(\mu-1)}{(1-\mu\sin^2\theta)^2}\left\{g\sin\theta(1-\mu\sin^2\theta) + 2g(\sin\theta - \sin\theta_0)\right\}$$

$$= \frac{m(\mu-1)}{(1-\mu\sin^2\theta)^2}\left\{3g\sin\theta - g\mu\sin^3\theta - 2g\sin\theta_0\right\}$$

$$= \frac{m(\mu-1)g}{(1-\mu\sin^2\theta)^2}\left\{3\sin\theta - \mu\sin^3\theta - 2\sin\theta_0\right\}$$

Alkraftinn $Q_r = \lambda\frac{\partial f}{\partial r} = \lambda \qquad \mu = \frac{m}{M+m}$

Ef $M \rightarrow \infty$, þ.e. $M >>> m$ fæst $\lambda = -mg\left\{3\sin\theta - 2\sin\theta_0\right\}$

Ef síðan $\theta_0 = 0 \rightarrow \lambda = -mg\,3\sin\theta$