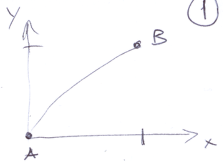
6-1

Milli punkta $(0,0)$ og $(1,1)$ í sléttu hnika

$$y(\alpha,x) = x + \alpha\cdot\sin[\pi(1-x)]$$



til að sjá að $y(0,x)=x$ sé stysti ferillinn (sjá mynd á næstu síðu).

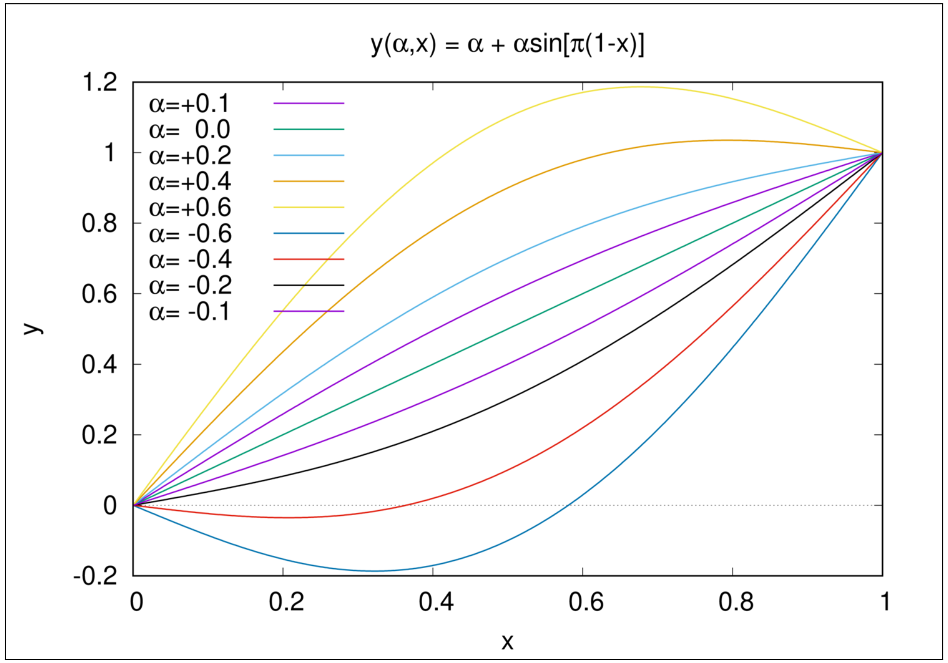
Lengd slóðarinnar er

$$S = \int_A^B \sqrt{dx^2+dy^2}$$

sem við útfærum með $y(\alpha,x)$ sem

$$S(\alpha) = \int_0^1 \sqrt{1+\left(\frac{dy(\alpha)}{dx}\right)^2}\,dx$$

$$\frac{dy(\alpha,x)}{dx} = 1-\alpha\pi\cos[\pi(1-x)]$$



$$S(\alpha) = \int_0^1 dx\sqrt{1+\{1-\alpha\pi\cos[\pi(1-x)]\}^2}$$

gerum breytuskipti $u=\pi(1-x)$

$$= \frac{1}{\pi}\int_0^\pi du\sqrt{1+\{1-\alpha\pi\cos(u)\}^2}$$

$$= \frac{\sqrt{2}}{\pi}\int_0^\pi du\sqrt{1-\alpha\pi\cos(u)+\frac{(\alpha\pi)^2}{2}\cos^2(u)}$$

Ellipskt heildi gerum ráð fyrir að $\alpha \ll 1$

$$\simeq \frac{\sqrt{2}}{\pi}\int_0^\pi du\left\{1-\frac{1}{2}\left(\alpha\pi\cos(u)-\frac{(\alpha\pi)^2}{2}\cos^2(u)\right) -\frac{1}{8}\left(\alpha\pi\cos(u)-\frac{(\alpha\pi)^2}{2}\cos^2(u)\right)^2+\dots\right\}$$

nú þarf að tína saman liði með minnst veldi $(\alpha\pi)^2$ og heilda

$$S(\alpha) \simeq \sqrt{2} + \frac{\sqrt{2}}{\pi}\int_0^\pi du\left\{-\frac{\alpha\pi}{2}\cos(u) + \frac{(\alpha\pi)^2}{4}\cos^2(u) - \frac{(\alpha\pi)^2}{8}\cos^2(u)\right\}+\dots$$

gefur 0

$$\simeq \sqrt{2} + \frac{\sqrt{2}}{\pi}\int_0^\pi du\,\frac{(\alpha\pi)^2}{8}\cos^2(u)+\dots \simeq \sqrt{2} + \frac{\sqrt{2}}{\pi}\frac{(\alpha\pi)^2}{8}\frac{\pi}{2}$$

$$= \sqrt{2} + \frac{\sqrt{2}}{16}(\pi\alpha)^2+\dots$$

$$\frac{\partial S(\alpha)}{\partial\alpha} = \frac{\sqrt{2}\pi^2\alpha}{8} \longrightarrow 0$$ þegar $\alpha = 0$

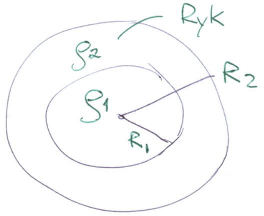
jafna fyrir beinalínu

Athugið að hér finnst lágmark þ. $\alpha=0$ og $\alpha \ll 1$

Við gætum misst af öðrum lágmörkum fyrir stærri α ...

5-13 (5)

Reikistjarna — Finna kraftinn á ryk ögn með massa m



Lögmál Gauß gildir

$$\oint \bar{g}\cdot d\bar{s} = -4\pi G M$$

Ef $\rho = \rho(r)$, óháð θ og ϕ þá er $\bar{g}$ einfalt $\bar{g} = g\hat{e}_r$

M gæti líka allt eins verið punktmassi, massinn utan r skiptir ekki máli

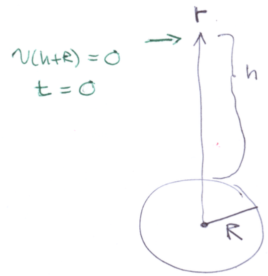
$$\longrightarrow \quad \bar{F}(r) = m\bar{g}(r) = -m\frac{GM(r)}{r^2}\hat{e}_r$$

$$M(r) = \underbrace{\frac{4\pi}{3}R_1^3\rho_1}_{\text{massi reikistjörnu}} + \underbrace{\frac{4\pi}{3}\left(r^3 - R_1^3\right)\rho_2}_{\text{massinn úr rykhjúpnum}}$$

(6)

$$\bar{F}(r) = -\frac{4\pi m}{3}G\left\{\frac{(\rho_1-\rho_2)R_1^3}{r^2} + \rho_2 r\right\}\hat{e}_r$$

05-05 Fall að jörð, sýna að fall um $\frac{h}{2}$ taki um það bil $\frac{9}{11}T$ þ.s. T er falltíminn



Hreyfijafnan er

$$m\ddot{r} = -G\frac{Mm}{r^2} \qquad (1)$$

Orkuvarðveisla

$$\frac{1}{2}m\dot{r}^2 - GM\frac{m}{r} = E = -GM\frac{m}{h+R} \qquad (2)$$

(7)

Endurritum (2)

$$\frac{1}{2}m\dot{r}^2 = GMm\overbrace{\left\{\frac{1}{r} - \frac{1}{R+h}\right\}}^{\geq 0} \qquad r \leq R+h$$

$$\longrightarrow \quad \dot{r} = \frac{dr}{dt} = -\sqrt{2GM\left\{\frac{1}{r} - \frac{1}{R+h}\right\}}$$

hraðinn er neikvæður niður á við

$$\longrightarrow \quad dt = -\frac{dr}{\sqrt{2GM\left\{\frac{1}{r} - \frac{1}{R+h}\right\}}}$$

heildum

$$\int_0^t dt' = -\int_{R+h}^{r}\frac{dr'}{\sqrt{2GM\left\{\frac{1}{r'} - \frac{1}{R+h}\right\}}}$$

(8)

$$t = -\frac{1}{\sqrt{GM2}}\int_{R+h}^{r} dr'\sqrt{\frac{r'(R+h)}{R+h-r'}}$$

Skiptum um breytu

$$y^2 = r' \longrightarrow 2y\,dy = dr', \longrightarrow \sqrt{r'}\,dr' = 2y^2dy$$

$$t = -\sqrt{\frac{2}{GM}}\int_{\sqrt{R+h}}^{\sqrt{r}} y^2dy\sqrt{\frac{R+h}{R+h-y^2}}$$

notum heildi úr viðbót E (E7)

$$= -\sqrt{R+h}\sqrt{\frac{2}{GM}}\left[-\frac{y}{2}\sqrt{R+h-y^2} + \frac{R+h}{2}\arcsin\left(\frac{y}{\sqrt{R+h}}\right)\right]_{\sqrt{R+h}}^{\sqrt{r}}$$

(9)

$$t = \frac{\sqrt{R+h}}{\sqrt{2GM}}\left\{\sqrt{r}\sqrt{R+h-r} - (R+h)\arcsin\left(\sqrt{\frac{r}{R+h}}\right) + (R+h)\frac{\pi}{2}\right\}$$

Fall úr $r = h+R$ í R

$$t(R) = \frac{\sqrt{R+h}}{\sqrt{2GM}}\left\{\sqrt{Rh} - (R+h)\arcsin\left(\sqrt{\frac{R}{R+h}}\right) + \frac{(R+h)\pi}{2}\right\}$$

Fall úr $r = h+R$ í $R+\frac{h}{2}$

$$t\left(R+\tfrac{h}{2}\right) = \frac{\sqrt{R+h}}{\sqrt{2GM}}\left\{\sqrt{R+\tfrac{h}{2}}\sqrt{\tfrac{h}{2}} - (R+h)\arcsin\left(\sqrt{\frac{R+\frac{h}{2}}{R+h}}\right) + \frac{(R+h)}{2}\pi\right\}$$

$h >> R$

$$t(R) = \frac{h^{3/2}\sqrt{1+\frac{R}{h}}}{\sqrt{2GM}}\left\{\sqrt{\frac{R}{h}} - \left(1+\frac{R}{h}\right)\arcsin\left(\sqrt{\frac{\frac{R}{h}}{1+\frac{R}{h}}}\right) + \frac{\left(1+\frac{R}{h}\right)\pi}{2}\right\}$$

(10)

$$t\left(R+\tfrac{h}{2}\right) = \frac{h^{3/2}\sqrt{1+\frac{R}{h}}}{\sqrt{2GM}}\left\{\frac{1}{2}\sqrt{1+\frac{2R}{h}} - \left(1+\frac{R}{h}\right)\arcsin\left(\sqrt{\frac{\frac{1}{2}+\frac{R}{h}}{1+\frac{R}{h}}}\right) + \frac{\left(1+\frac{R}{h}\right)\pi}{2}\right\}$$

$h >> R$, geymum lægstu nálgun

$$t(R) \simeq \frac{h^{3/2}}{\sqrt{2GM}}\,\frac{\pi}{2}\,, \qquad t\left(R+\tfrac{h}{2}\right) \simeq \frac{h^{3/2}}{\sqrt{2GM}}\left\{\frac{1}{2} - \arcsin\left(\tfrac{1}{2}\right) + \frac{\pi}{2}\right\}$$

$$= \frac{h^{3/2}}{\sqrt{2GM}}\left\{\frac{1}{2} - \frac{\pi}{6} + \frac{\pi}{2}\right\}$$

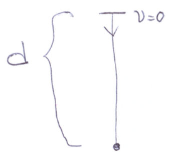
$$\rightarrow \quad \frac{t\left(R+\frac{h}{2}\right)}{t(R)} = \frac{\left(\frac{3-\pi+3\pi}{6}\right)}{\pi/2}$$

$$= \frac{2}{6}\left(\frac{3+2\pi}{\pi}\right) = \frac{1}{3\pi}(3+2\pi)$$

$$= \frac{2\pi}{3\pi}\left(1+\frac{3}{2\pi}\right) = \frac{2}{3}\left(1+\frac{3}{2\pi}\right) = 0.985$$

(11)

05-04



$$F = -\frac{mk^2}{r^3}\,, \quad \text{Sýna að} \quad T_d = \frac{d^2}{k}$$

$\bar{F} = -\bar{\nabla} U$ því getum við valið $U(r) = -\frac{mk^2}{2r^2}$

Miðlægt mætti og geymnir kraftar

Orkuvarðveisla

$$E = \frac{m}{2}\dot{r}^2 - \frac{mk^2}{2r^2} = -\frac{mk^2}{2d^2}$$

Ef upphafsorkan er eingöngu mættisorka, $v(d) = 0$

eða $v(t=0) = 0$

$$\frac{m\dot{r}^2}{2} = \frac{mk^2}{2}\left\{\frac{1}{r^2} - \frac{1}{d^2}\right\}$$

$$\left(\frac{dr}{dt}\right)^2 = k^2\left\{\frac{1}{r^2} - \frac{1}{d^2}\right\} \quad \rightarrow \quad dt = \frac{dr}{k\sqrt{\frac{1}{r^2} - \frac{1}{d^2}}}$$

(12)

$$\int_0^t dt' = -\int_d^0 \frac{dr}{k\sqrt{\frac{1}{r^2} - \frac{1}{d^2}}} = -\frac{d}{k}\int_d^0 \frac{r\,dr}{\sqrt{d^2-r^2}}$$

$$= +\frac{d}{k}\sqrt{d^2-r^2}\,\Big|_d^0 = \frac{d^2}{k} = T_d$$

formerkið á rótinni er valið til að t vaxi þegar r minnkar

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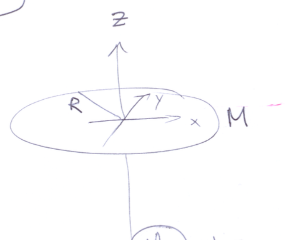
5-20

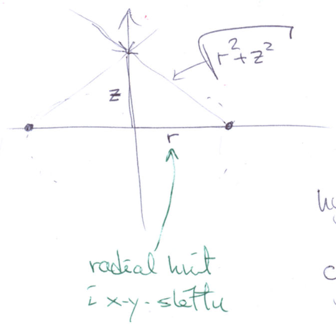
Reikna $\Phi(z)$ og $\bar{g}(z)$

á z-ás



Athugum mættið frá grönnum hring

Einfaldast er að staða Ex. 5.4 í bók og setja saman Φ beint



$$\boxed{d\Phi = -G\frac{dM}{\sqrt{r^2+z^2}}}$$

hugsum $dM = \rho^{2D} dA = \rho^{2D}\, 2\pi r\, dr$

og $\rho^{2D} = \frac{M}{\pi R^2}$ ← jöfnuð massadreifing

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$$dM = \left(\frac{M}{\pi R^2}\right) 2\pi r\, dr = 2M\frac{r\, dr}{R^2}$$ ← gott að sjá rétta vídd

$$\to d\Phi = -G\frac{2M\, r\, dr}{R^2\sqrt{r^2+z^2}}$$

$$\Phi(z) = -\frac{G2M}{R^2}\int_0^R \frac{r\, dr}{\sqrt{r^2+z^2}} = -\frac{GM2}{R^2}\sqrt{r^2+z^2}\Big|_0^R$$

$$= -\frac{2GM}{R^2}\left\{\sqrt{R^2+z^2} - z\right\}$$

Hér höfum við vísvitandi ekki tilgreint viðmiðunarpunkt mættis.

$$\bar{g}(z) = -\hat{e}_z\frac{d\Phi}{dz} = +\frac{2GM}{R^2}\hat{e}_z\left\{\frac{z}{\sqrt{R^2+z^2}} - 1\right\}$$

$$= -\frac{2GM}{R^2}\hat{e}_z\left\{\frac{\sqrt{R^2+z^2} - z}{\sqrt{R^2+z^2}}\right\}$$

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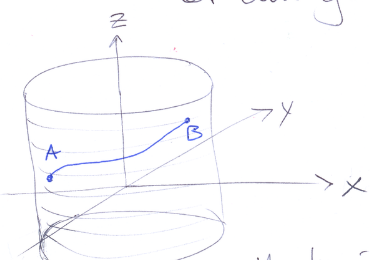
Ef $R\to\infty$ þarf að muna að $\frac{M}{R^2}$ er fasti og í ljós kemur að $\bar{g}$ verður óháð z

Enginn lengdarstuðli er eftir í kerfinu til að miða z við, og krafturinn er allstaðar í sömu átt

06-04

Stysta leið milli tveggja punkta á sívalningi er um gorm feril

Geisli fastur $\rho = a$



Í kartísum hnitum er leiðarfrymið

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Hentugri eru sívalningshnit

$x = a\cos\phi$ $\quad z = z$

$y = a\sin\phi$

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$dx = -a\sin\phi \cdot d\phi$ $\quad dz = dz$

$dy = +a\cos\phi \cdot d\phi$

$$\to ds = \sqrt{(a\,d\phi)^2 + (dz)^2} = \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}\cdot d\phi$$

þurfum að lágmarka

$$S = \int_A^B d\phi\sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2} = \int_A^B d\phi\, f\left(z, \frac{dz}{d\phi}\right)$$

en $f = f\left(\frac{dz}{d\phi}\right)$ og z er fast í a

Athugið að heildarafleiðan $\frac{dz}{d\phi}$ lýsir því hvernig z breytist á leiðinni þegar ϕ breytist. $\frac{\partial z}{\partial \phi} = 0$, en $\frac{dz}{d\phi} \neq 0$

Höfum engar upplýsingar um $\frac{dz}{d\phi}$, þar sem við höfum enga stikun fyrir leiðina

→ notum að S lágmarkast þegar

$$\underbrace{\frac{\partial f}{\partial z}}_{=0} - \frac{\partial}{\partial \phi} \frac{\partial f}{\partial \left(\frac{dz}{d\phi}\right)} = 0$$

jafna Eulers

$$\rightarrow \frac{\partial}{\partial \phi} \frac{\left(\frac{dz}{d\phi}\right)}{\sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}} = 0 \quad \rightarrow \frac{\left(\frac{dz}{d\phi}\right)}{\sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}} = \text{fasti} = C$$

$$\rightarrow \underbrace{\left(\frac{dz}{d\phi}\right) = \sqrt{\frac{C^2}{1-C^2}}\, a = \text{fasti}}_{}$$

jafnan fyrir einsleitum gormi