3-15

$$x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$$

$$\dot{x}(t) = -A e^{-\beta t} \left\{ \beta \cos(\omega_1 t - \delta) + \omega_1 \sin(\omega_1 t - \delta) \right\}$$

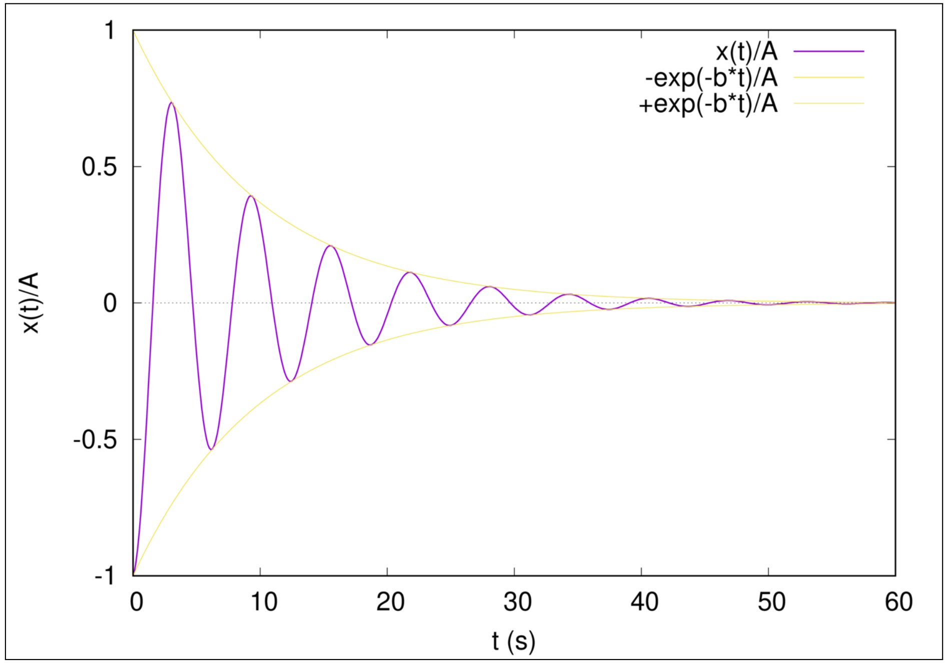
$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

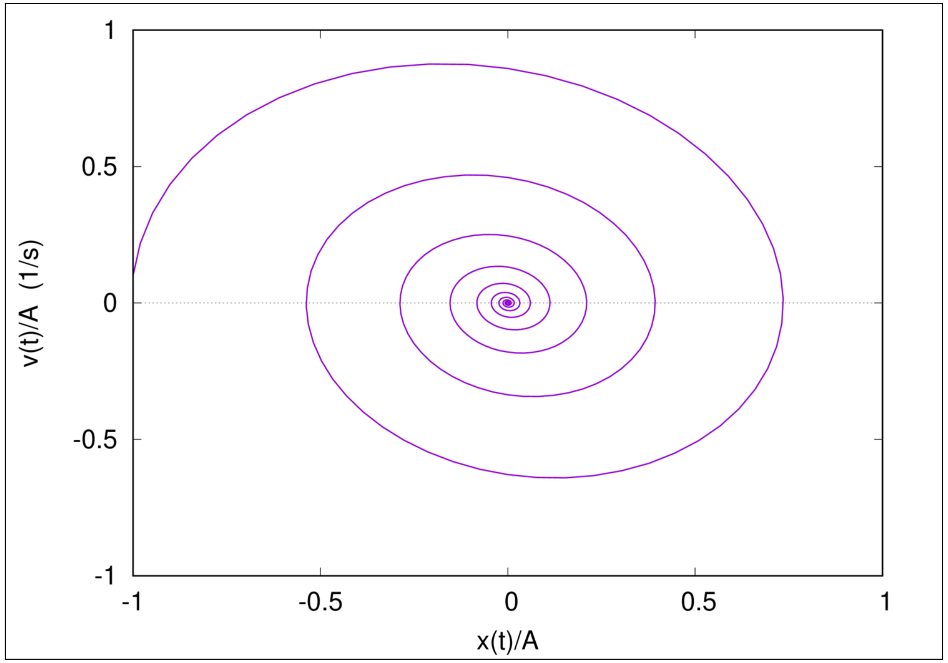
$A = 1$, $\omega_0 = 1$ rad/s

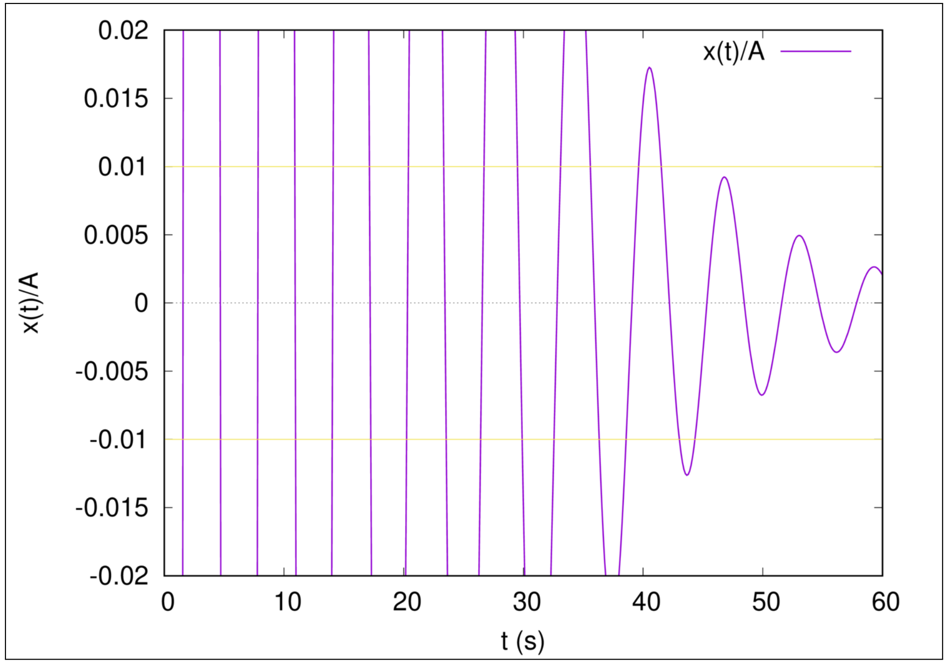
$\beta = 0{,}1\ s^{-1}$ $\quad \delta = \pi$ rad

Hve margar sveiflur áður en $\frac{x}{A} < 10^{-2}$

sjá næstu 3 gröf þar sem $t \in [0, 60]$ s







3-42 Ódeyfður hreintóna sveifill

5

$$m\ddot{x} + m\omega_0^2 x = \Theta(t) F_0 \sin(\omega t)$$

upphafsskilyrði

$x(0) = 0$
$v(0) = 0$

Finna lausn þegar $\omega \to \omega_0$

Almenn lausn óhliðruðu jöfnunnar er

$$x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

ein sérstök lausn

$$C\sin(\omega t) \to \left(-m\omega^2 + m\omega_0^2\right)C = F_0 \quad (*)$$

$$\to \quad x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) + C\sin(\omega t)$$

$$0 = B\cos(\omega_0 t) \to B = 0$$

$$0 = A\omega_0 + C\omega \to$$

6

$$(*) \to C = \frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{og því} \quad A\omega_0 = -\frac{F_0\omega}{m(\omega_0^2-\omega^2)}$$

$$A = -\frac{F_0\omega}{m\omega_0}\frac{1}{(\omega_0^2-\omega^2)}$$

$$\to x(t) = -\frac{F_0\omega}{m\omega_0}\frac{\sin(\omega_0 t)}{(\omega_0^2-\omega^2)} + \frac{F_0\sin(\omega t)}{m(\omega_0^2-\omega^2)}$$

$$= \frac{F_0}{m\omega_0}\frac{1}{(\omega_0^2-\omega^2)}\left\{\omega_0\sin(\omega t) - \omega\sin(\omega_0 t)\right\}$$

b)

$$x(t) = \frac{F_0}{m\omega_0(\omega_0+\omega)}\left\{\frac{\omega_0\sin(\omega t) - \omega\sin(\omega_0 t)}{\omega_0 - \omega}\right\}$$

7

$$\lim_{\omega\to\omega_0}\left\{\frac{\omega_0\sin(\omega t) - \omega\sin(\omega_0 t)}{\omega_0-\omega}\right\} = \sin(\omega_0 t) - (\omega_0 t)\cos(\omega_0 t)$$

$\to$ fyrir $\omega \to \omega_0$

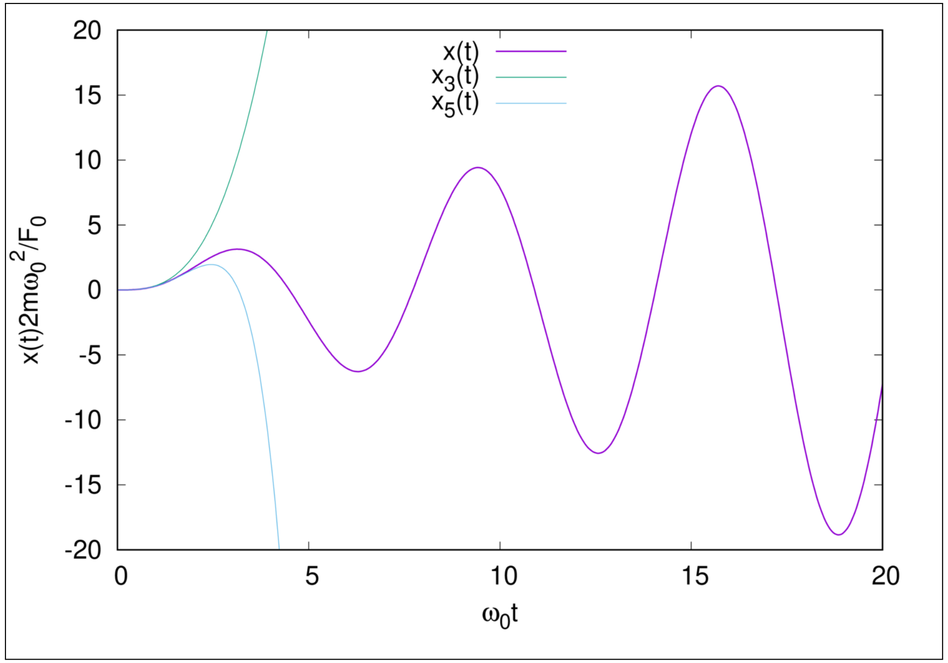
$$x(t) = \frac{F_0}{2m\omega_0^2}\left\{\sin(\omega_0 t) - (\omega_0 t)\cos(\omega_0 t)\right\}$$

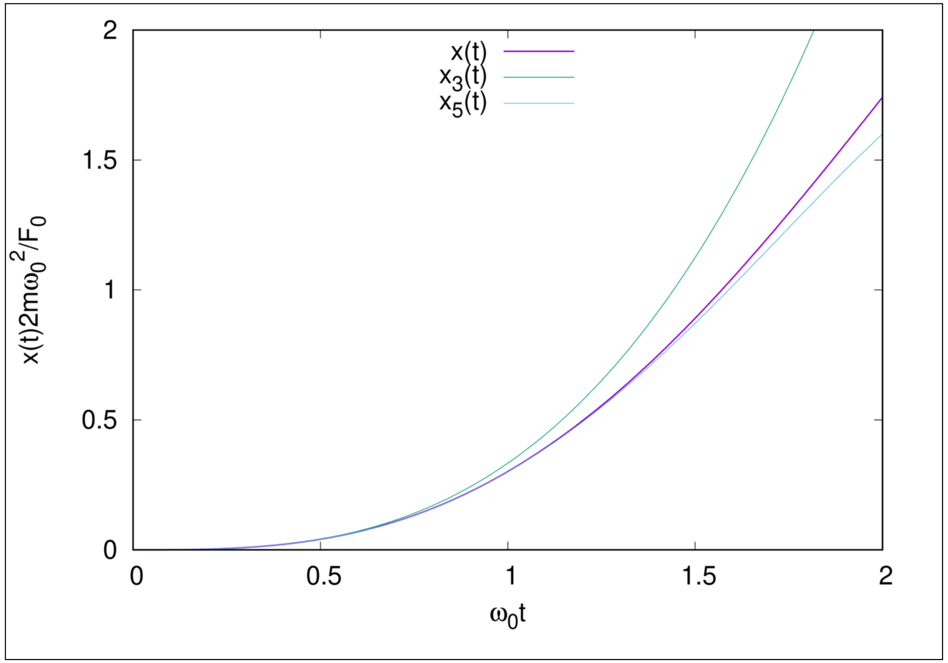
Síðan má sjá

$$x(t) = \frac{F_0}{2m\omega_0^2}\left\{\frac{(\omega_0 t)^3}{3} - \frac{(\omega_0 t)^5}{30} + \cdots\right\}$$

þegar $(\omega_0 t) \to 0$

$x_3(t)$ $x_5(t)$

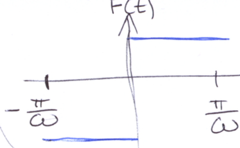




3-28

10

Fourier röð fyrir



$$F(t) = \begin{cases} -1 & -\frac{\pi}{\omega} < t < 0 \\ +1 & 0 < t < \frac{\pi}{\omega} \end{cases}$$

Oddstætt fall

$$F(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} dt'\, F(t')\sin(n\omega t) \qquad n$$

$$= \frac{\omega}{\pi}\left[-\int_{-\pi/\omega}^{0} dt' \sin(n\omega t) + \int_{0}^{\pi/\omega} dt' \sin(n\omega t)\right]$$

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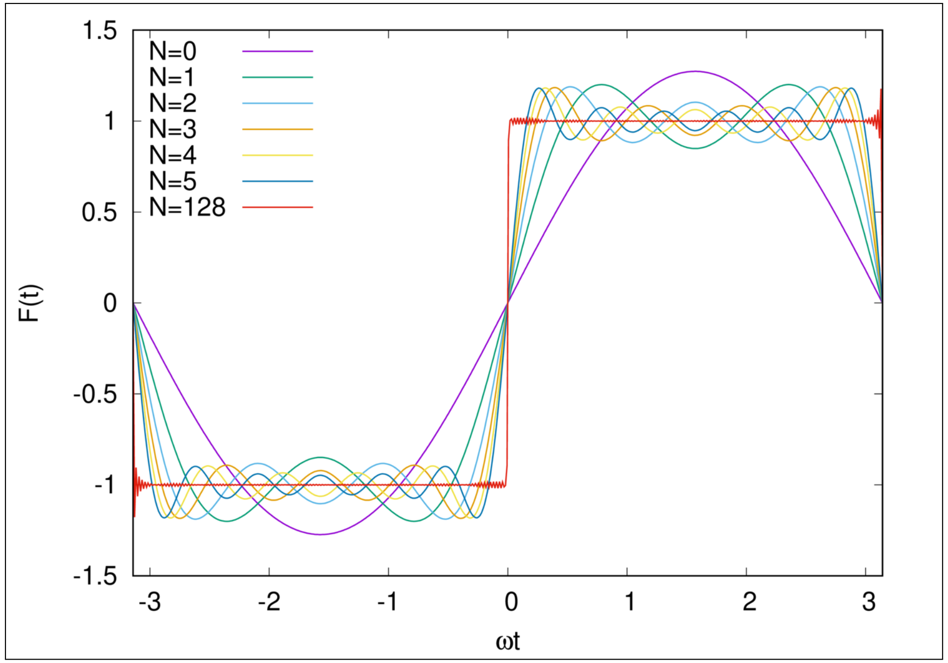
$$= \frac{\omega}{\pi}\left\{ \frac{\cos(n\omega t')}{n\omega}\Big|_{-\pi/\omega}^{0} + \left(-\frac{\cos(n\omega t')}{n\omega}\Big|_{0}^{\frac{\pi}{\omega}}\right)\right\}$$

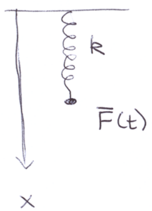
$$= -\frac{\omega}{\pi}\left\{2\frac{\cos(n\pi)}{n\omega} - 2\frac{\cos(0)}{n\omega}\right\}$$

$$= \frac{2}{\pi n}\left\{1 - \cos(n\pi)\right\} = \begin{cases} \frac{4}{n\pi} & n \in \text{oddatala} \\ 0 & n \text{ jöfn} \end{cases}$$

$$\rightarrow \quad F(t) = \frac{4}{\pi}\sum_{n=0,1,2,\dots}^{\infty} \frac{\sin((2n+1)\omega t)}{(2n+1)}$$



03-09



$$\bar{F}(t) = \theta(t)\,\theta(t_0 - t)\,F\,\hat{x}$$

$0 < t < t_0$

$$m\ddot{x} = -k(x - x_0) + F$$

$t > t_0$

$$m\ddot{x} = -k(x - x_0)$$

Sýna að eftir t_0 sé lausnin

$$x - x_0 = \frac{F}{k}\left\{\cos(\omega_0(t - t_0)) - \cos(\omega_0 t)\right\}$$

með $\omega_0^2 = \frac{k}{m}$

Skipta um breytu $y = x - x_0$

→ Hreyfijafnan verður

$$\begin{cases} m\ddot{y} = -ky + F & \text{þ. } 0 < t < t_0 \quad (1) \\ m\ddot{y} = -ky & \text{þ. } t > t_0 \quad (2) \end{cases}$$

Lausn (2) er

$$y(t) = Ae^{i\omega t} + Be^{-i\omega t}, \quad \omega = \sqrt{\frac{k}{m}} \qquad t > t_0$$

Lausn (1) er

$$y(t) = Ce^{i\omega t} + De^{-i\omega t} + \frac{F}{k} \qquad 0 < t < t_0$$

→ því sem síðasti liðurinn er sérlausn á hliðruðu jöfnunni (1)

Veljum upphafsskilyrði $y(0) = 0$, $\dot{y}(0) = 0$ ⇐ F(t) kemur kerfinu í gang....

$$\left.\begin{aligned} y(0) = 0 &\to C + D + \frac{F}{k} = 0 \\ \dot{y}(0) = 0 &\to i\omega C - i\omega D = 0 \end{aligned}\right\} \to C = D = -\frac{F}{2k}$$

Lausnin er samfelld í t_0, bæði y og $\dot{y}$

$y(t_0^-) = y(t_0^+)$

$$-\frac{F}{2k}e^{i\omega t_0} - \frac{F}{2k}e^{-i\omega t_0} + \frac{F}{k} = Ae^{i\omega t_0} + Be^{-i\omega t_0}$$

$\dot{y}(t_0^-) = \dot{y}(t_0^+)$

$$-\frac{F}{2k}i\omega e^{i\omega t_0} + \frac{F}{2k}i\omega e^{-i\omega t_0} = i\omega A e^{i\omega t_0} - i\omega B e^{-i\omega t_0}$$

$$\frac{F}{k}\left\{1 - \cos(\omega t_0)\right\} = Ae^{i\omega t_0} + Be^{-i\omega t_0}$$

$$-\frac{F}{k}\sin(\omega t_0) = Ae^{i\omega t_0} - Be^{-i\omega t_0}$$

$$\begin{pmatrix} e^{i\omega t_0} & e^{-i\omega t_0} \\ e^{i\omega t_0} & -e^{-i\omega t_0} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \frac{F}{k}\begin{pmatrix} 1 - \cos(\omega t_0) \\ -\sin(\omega t_0) \end{pmatrix}$$

$$\to \boxed{\begin{aligned} A &= \frac{F}{2k}e^{-i\omega t_0}\left(1 - e^{+i\omega t_0}\right) \\ B &= \frac{F}{2k}e^{+i\omega t_0}\left(1 - e^{-i\omega t_0}\right) \end{aligned}}$$

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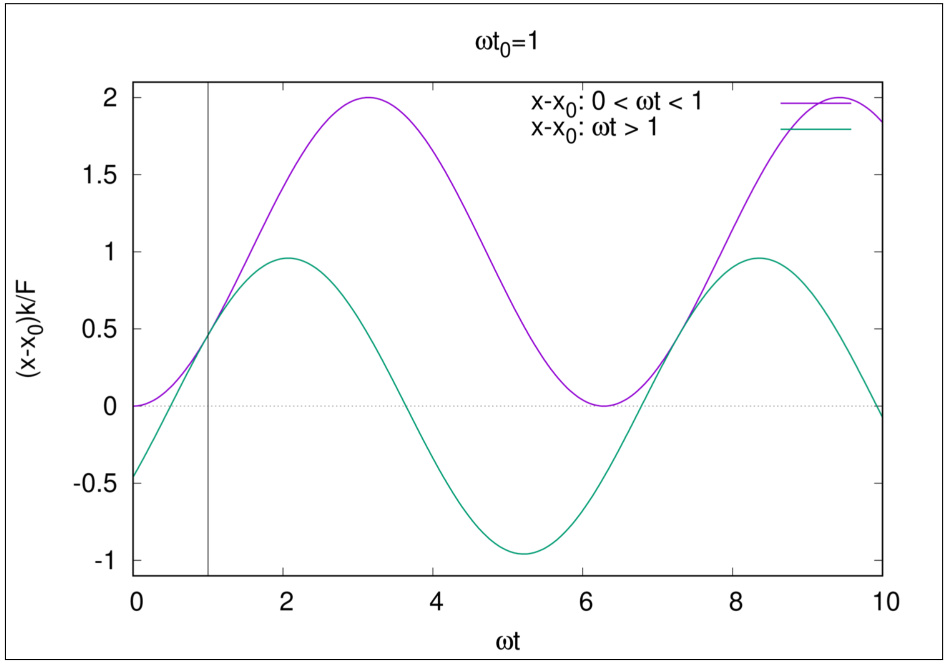
Lausnin

$$y(t) = \frac{F}{k}\left\{1 - \frac{e^{i\omega t}}{2} - \frac{e^{-i\omega t}}{2}\right\}$$

$$= \frac{F}{k}\left\{1 - \cos(\omega t)\right\}, \qquad 0 < t < t_0$$

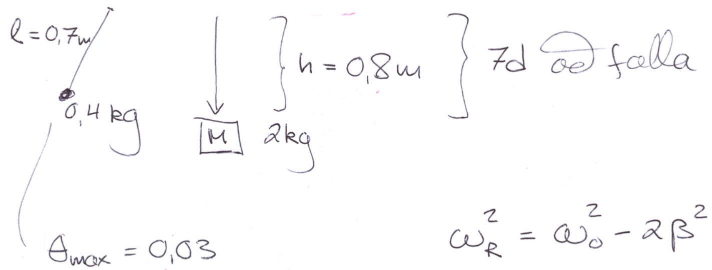
$$y(t) = \frac{F}{2k}\left\{e^{i\omega(t-t_0)}\left(1-e^{i\omega t_0}\right) + e^{-i\omega(t-t_0)}\left(1-e^{-i\omega t_0}\right)\right\}$$

$$= \frac{F}{k}\left\{\cos(\omega(t-t_0)) - \cos(\omega t)\right\}, \qquad t > t_0$$



19

3-45 Finna Q fyrir klukku, $Q = \frac{\omega_R}{2\beta}$



$\theta_{max} = 0{,}03$

$\omega_R^2 = \omega_0^2 - 2\beta^2, \quad \omega_0 = \sqrt{g/l}$

Fyrir veika deyfingu er lausnin fyrir sveifluna

$\theta(t) = \theta_{max} \cdot e^{-\beta t}$ þ.s. β er dofnunarstuðullinn

úr hreyfijöfnunni

$$\ddot{\theta} + 2\sqrt{g/l}\,\dot{\theta} + \frac{g}{l}\theta = 0$$

hér vantar í raun lið sem bætir við orku

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Þurfum að meta max gildi á β sem er þannig að orkan út úr kerfinu jafnist á við stöðuorku lóðs

$$E_{pot} = Mgh$$

Sem nægir í 7d t.a. halda því gangandi, $\tau = 7d$

Upphafsorka pendúls

$$E_p(0) = \frac{1}{2} mgl\theta^2$$

Lota hans er $T = 2\pi\sqrt{\frac{l}{g}}$

$$\rightarrow E_p(T) = \frac{1}{2} mgl\,\theta^2(T) = \frac{1}{2} mgl\,\theta_{max}^2 e^{-2\beta T}$$

því er orkan týnd í einni sveiflu

$$E_p(T) - E_p(0) = \frac{1}{2} mgl\,\theta_{max}^2\left\{1 - e^{-2\beta T}\right\}$$

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$$E_p(T) - E_p(0) \simeq m g l \theta_{max}^2 \beta T$$

þess vegna fyrir tíma bilið τ

$$m g l \theta_{max}^2 \beta T \left(\frac{\tau}{T}\right) = m g l \theta_{max}^2 \beta \tau$$

sú orka verður að vera jöfn stöðuorku M

$$\rightarrow \quad m g l \theta_{max}^2 \beta \tau = M g h$$

$$\rightarrow \quad \beta = \frac{M h}{m l \theta_{max}^2 \tau}$$

$$Q = \frac{\omega_R}{2\beta} = \frac{\sqrt{\frac{g}{l} - 2\beta^2}}{2\beta} = \frac{1}{2}\sqrt{\frac{g}{l\beta^2} - 2}$$

$$= \frac{1}{2}\sqrt{\frac{g m^2 l \theta_{max}^4 \tau^2}{M^2 h^2} - 2} \simeq \underline{178.3}$$

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03-23

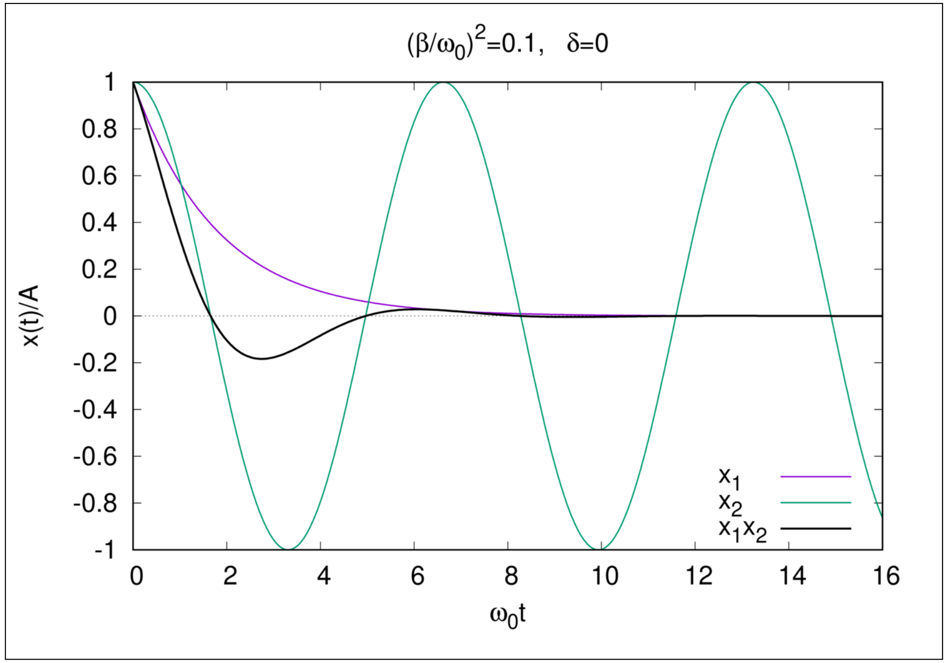
$$x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$$

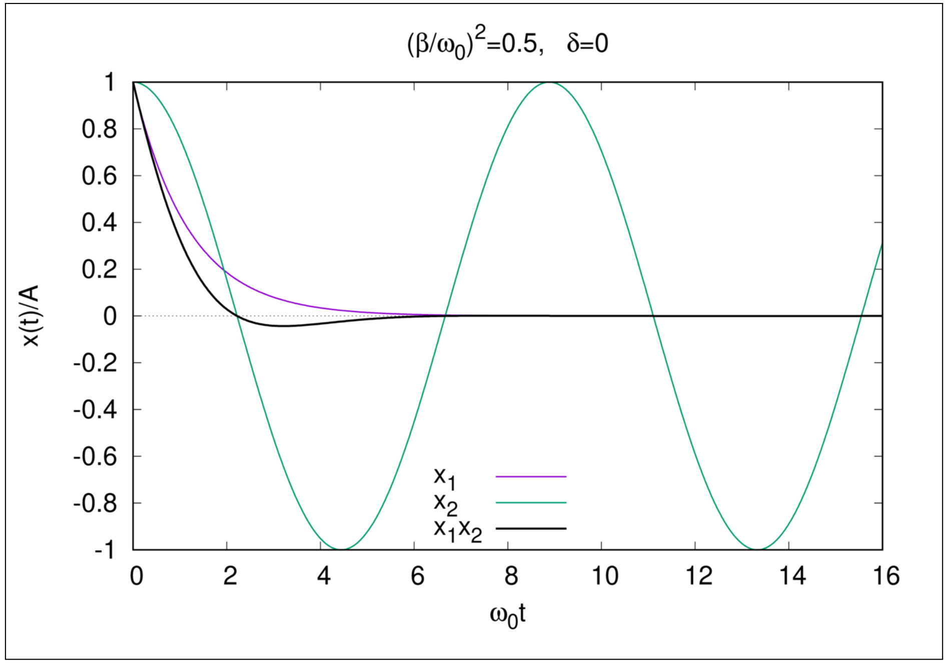
$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \omega_0 \sqrt{1 - \frac{\beta^2}{\omega_0^2}}$$

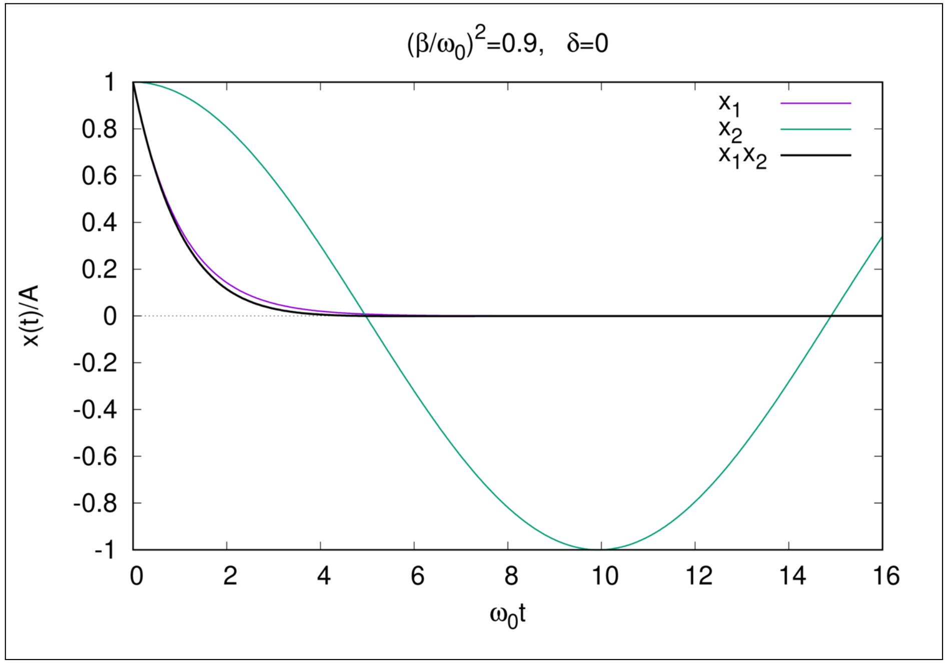
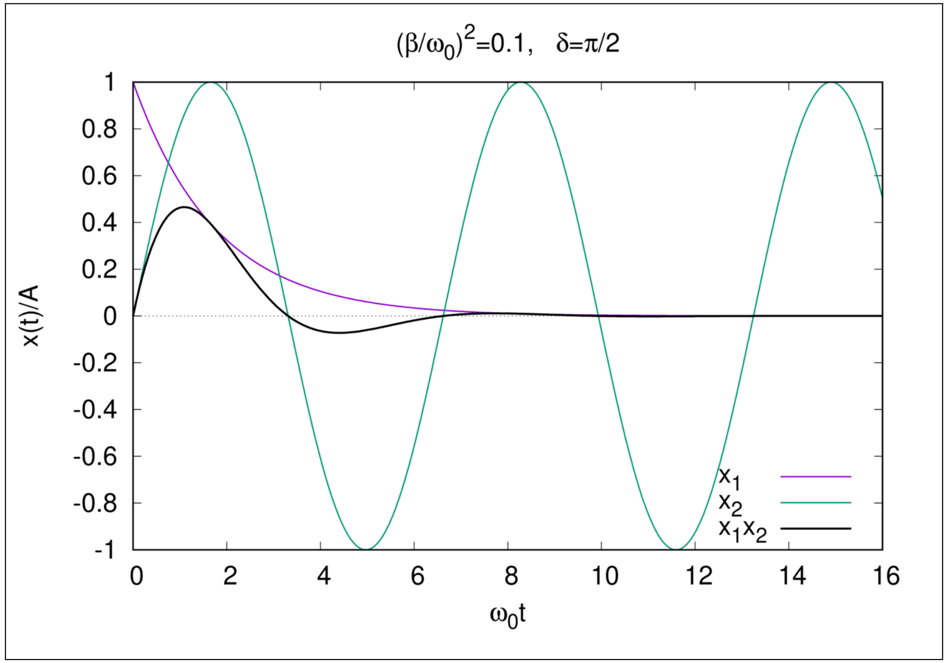
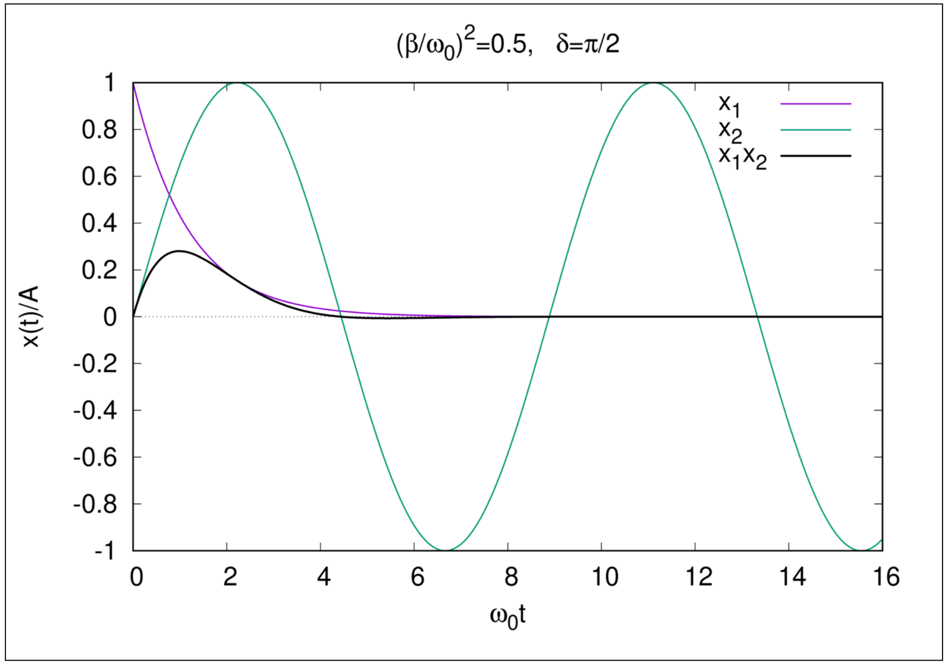
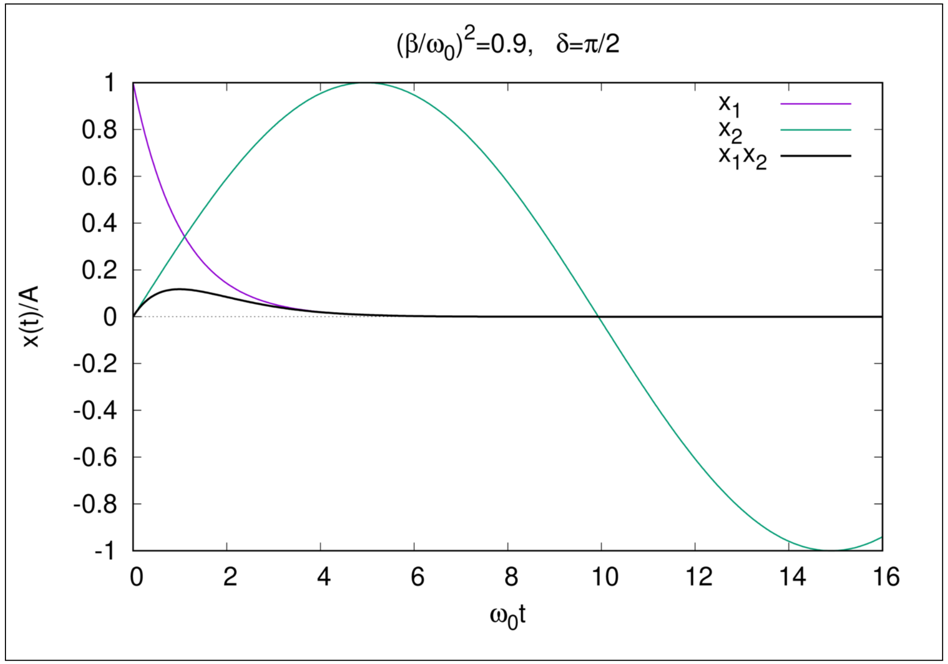
$$\frac{\beta^2}{\omega_0^2} = 0.1, \; 0.5, \; 0.9 \quad \Big| \quad \delta = 0, \; \frac{\pi}{2}, \; \pi$$

$$\frac{x(t)}{A} = \underbrace{\exp\left\{-\left(\frac{\beta}{\omega_0}\right)(t\omega_0)\right\}}_{x_1(t)} \underbrace{\cos\left\{(t\omega_0)\sqrt{1-\left(\frac{\beta}{\omega_0}\right)^2} - \delta\right\}}_{x_2(t)}$$

Gröfin fylgja






(β/ω0)2=0.9,   δ=0
x1
x2
x1x2
x(t)/A
ω0t
1 0.8 0.6 0.4 0.2 0 -0.2 -0.4 -0.6 -0.8 -1
0 2 4 6 8 10 12 14 16

(β/ω0)2=0.1,   δ=π/2
x1
x2
x1x2
x(t)/A
ω0t
1 0.8 0.6 0.4 0.2 0 -0.2 -0.4 -0.6 -0.8 -1
0 2 4 6 8 10 12 14 16

(β/ω0)2=0.5,   δ=π/2
x1
x2
x1x2
x(t)/A
ω0t
1 0.8 0.6 0.4 0.2 0 -0.2 -0.4 -0.6 -0.8 -1
0 2 4 6 8 10 12 14 16

(β/ω0)2=0.9,   δ=π/2
x1
x2
x1x2
x(t)/A
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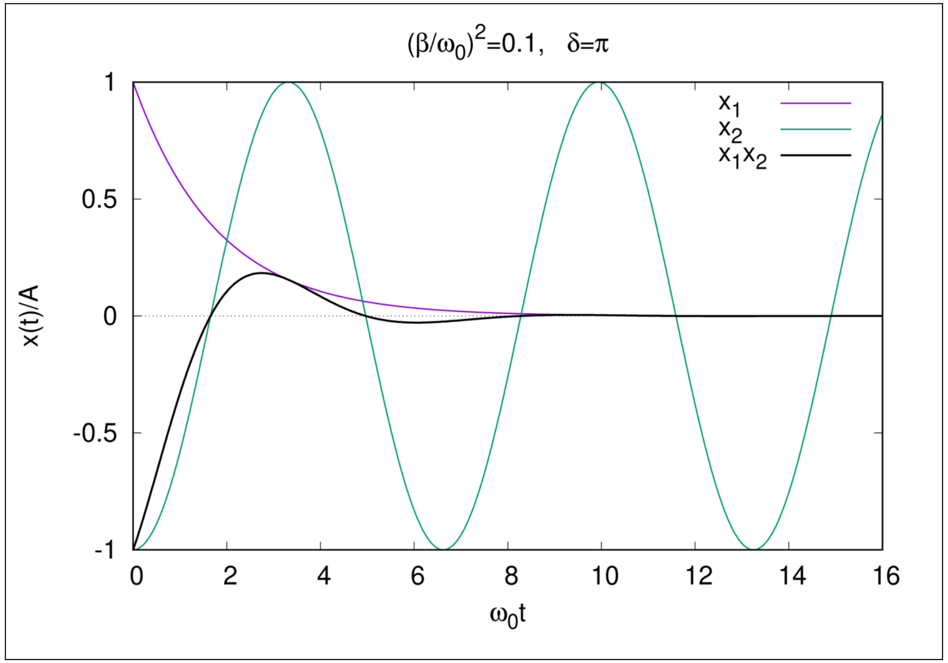
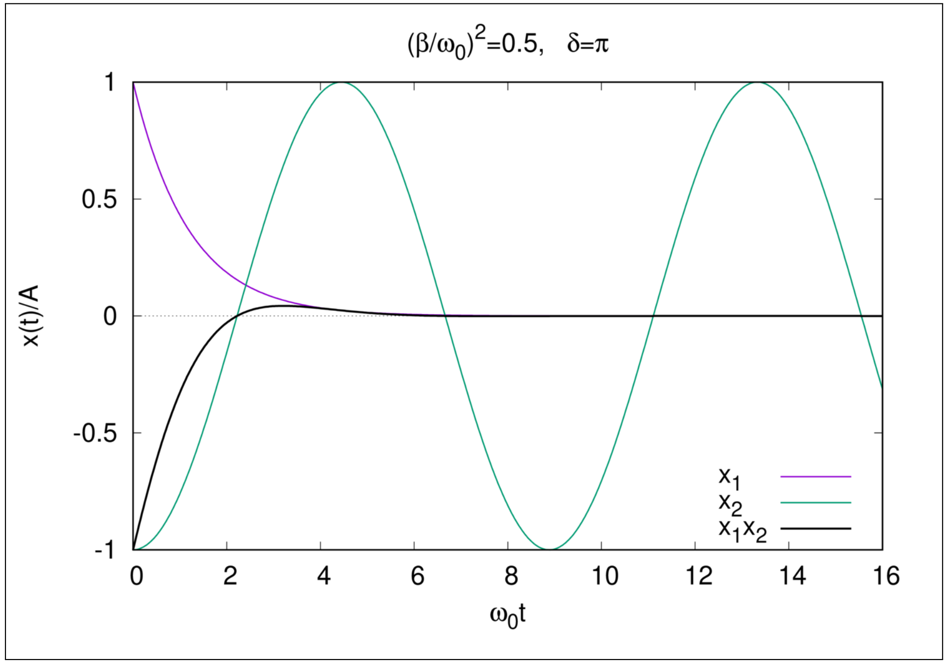
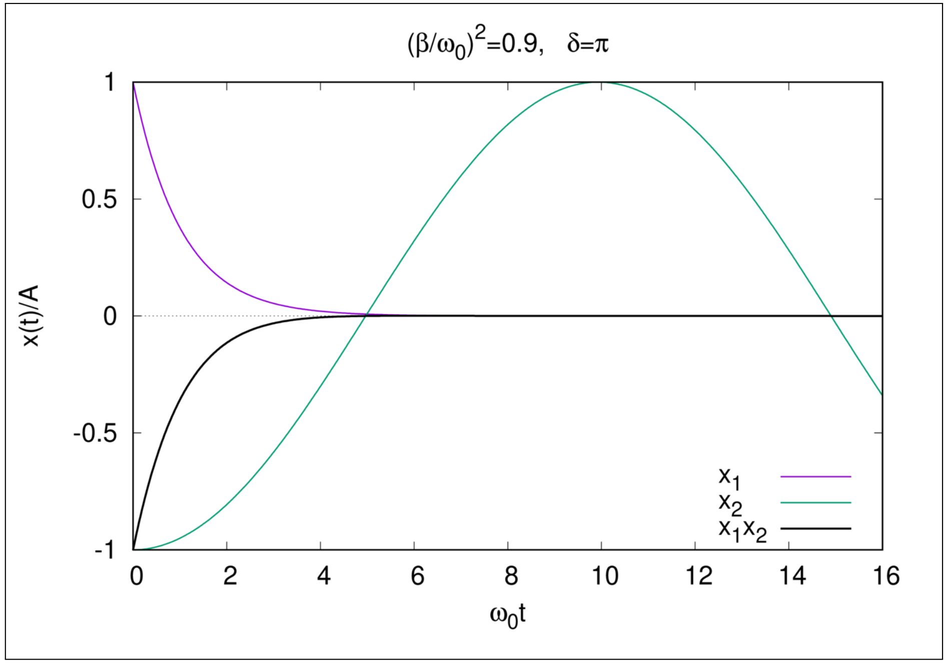
1 0.8 0.6 0.4 0.2 0 -0.2 -0.4 -0.6 -0.8 -1
0 2 4 6 8 10 12 14 16



$(\beta/\omega_0)^2$=0.1, δ=π
x(t)/A
ω_0t
x_1
x_2
x_1x_2
1
0.5
0
-0.5
-1
0 2 4 6 8 10 12 14 16



$(\beta/\omega_0)^2$=0.5, δ=π
x(t)/A
ω_0t
x_1
x_2
x_1x_2
1
0.5
0
-0.5
-1
0 2 4 6 8 10 12 14 16



$(\beta/\omega_0)^2$=0.9, δ=π
x(t)/A
ω_0t
x_1
x_2
x_1x_2
1
0.5
0
-0.5
-1
0 2 4 6 8 10 12 14 16