(1)

$$m\ddot{x} = -\lambda \,\mathrm{sgn}(\dot{x})\,|\dot{x}|^{\alpha}$$

$t>0,\quad \dot{x}>0$ gerum ráð fyrir

$\dot{x}(t=0) = v_0 \qquad x(t=0) = x_0$

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{m}\left(\frac{dx}{dt}\right)^{\alpha}$$

Breytum í jöfnu fyrir v

$$\frac{dv}{dt} = -\frac{\lambda}{m} v^{\alpha}$$

$$\frac{dv}{v^{\alpha}} = -\frac{\lambda}{m}\,dt$$

heildum

$$\int_{v_0}^{v} \frac{dv'}{(v')^{\alpha}} = -\frac{\lambda}{m}\int_0^t dt$$

(1)

$$-\frac{v^{1-\alpha}}{\alpha-1} + \frac{v_0^{1-\alpha}}{\alpha-1} = -\frac{\lambda}{m} t$$

$$v^{1-\alpha} - v_0^{1-\alpha} = \frac{\lambda}{m}(\alpha-1)t$$

$$v^{1-\alpha} = v_0^{1-\alpha} + \frac{\lambda}{m}(\alpha-1)t$$

$$= (\alpha-1)\left(-\frac{v_0^{1-\alpha}}{1-\alpha} + \frac{\lambda t}{m}\right)$$

$$\boxed{v = \left\{(\alpha-1)\left(-\frac{v_0^{1-\alpha}}{1-\alpha} + \frac{\lambda t}{m}\right)\right\}^{\frac{1}{1-\alpha}}}$$

$$\frac{dx}{dt} = \left\{(\alpha-1)\left(-\frac{v_0^{1-\alpha}}{1-\alpha} + \frac{\lambda t}{m}\right)\right\}^{\frac{1}{1-\alpha}}$$

(2)

$$\int_{x_0}^{x} dx' = \int_0^t dt' \left\{(\alpha-1)\left(-\frac{v_0^{1-\alpha}}{1-\alpha} + \frac{\lambda t'}{m}\right)\right\}^{\frac{1}{1-\alpha}}$$

heildum

$$x - x_0 = \frac{m}{\lambda}\frac{(\alpha-1)^{\frac{1}{1-\alpha}}}{\left(\frac{2-\alpha}{1-\alpha}\right)}\left(-\frac{v_0^{1-\alpha}}{1-\alpha} + \frac{\lambda t'}{m}\right)^{\frac{1+1-\alpha}{1-\alpha}}\Bigg|_0^t \qquad \frac{2-\alpha}{1-\alpha}$$

$$= -\frac{m}{\lambda(2-\alpha)}\left\{(\alpha-1)\left(-\frac{v_0^{1-\alpha}}{1-\alpha} + \frac{\lambda t'}{m}\right)\right\}^{\frac{2-\alpha}{1-\alpha}}\Bigg|_0^t$$

$$= +\frac{m}{\lambda}\frac{v_0^{2-\alpha}}{(2-\alpha)} - \frac{m}{\lambda(2-\alpha)}\left\{(\alpha-1)\left(\frac{v_0^{1-\alpha}}{(\alpha-1)} + \frac{\lambda t}{m}\right)\right\}^{\frac{2-\alpha}{1-\alpha}}$$

$$= \frac{m}{\lambda(2-\alpha)}\left[v_0^{2-\alpha} - \left\{v_0^{1-\alpha} + \frac{\lambda t}{m}(\alpha-1)\right\}^{\frac{2-\alpha}{1-\alpha}}\right]$$

(3)

Því fæst

$$x - x_0 = \frac{m v_0^{2-\alpha}}{\lambda(2-\alpha)}\left[1 - \left\{1 + \frac{\lambda t(\alpha-1)}{m v_0^{1-\alpha}}\right\}^{\frac{2-\alpha}{1-\alpha}}\right]$$

nálgum þegar t er lítið, þ.e. ef $\dfrac{\lambda t(\alpha-1)}{m v_0^{1-\alpha}} \ll 1$

(við þurfum alltaf víddir, sem fæst með því að gera breyturnar víddarlausar)

notum Taylor $\quad (1+x)^{\beta} \simeq 1 + \beta x + \dfrac{(\beta^2-\beta)}{2}x^2 + \ldots\ldots$

$$x - x_0 \simeq \frac{m v_0^{2-\alpha}}{\lambda(2-\alpha)}\left\{-\frac{(2-\alpha)}{(1-\alpha)}\frac{\lambda t(\alpha-1)}{m v_0^{1-\alpha}} - \frac{\left(\left(\frac{2-\alpha}{1-\alpha}\right)^2 - \frac{(2-\alpha)}{(1-\alpha)}\right)}{2}\frac{\lambda^2 t^2(\alpha-1)^2}{m^2 v_0^{2(1-\alpha)}} + \ldots\right\}$$

$$\simeq v_0 t - \frac{\lambda}{2m}v_0^{\alpha}t^2 + \ldots$$

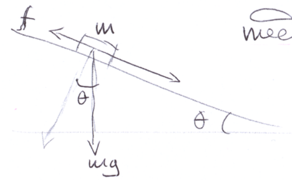
fyrir lítið t er eins og kraftur λv_0^{α} sé á móti hreyfingunni

sjá M. Razavy, Dissipative classical and Quantum systems 2nd Ed. 1. kafla

(4)

2-15

Skoðum, hreyfing agnar í föstu þyngdarsviði og með viðnámskraft $f = kmv^2$



Hve langan tíma þarf ögnin til að komast d úr kyrrstöðu?

Hreyfijafnan er

$$m\ddot{x} = mg\sin\theta - kmv^2 \;\longrightarrow\; \boxed{\frac{dv}{dt} = g\sin\theta - kv^2} \quad (*)$$

eða

$$dv = g\sin\theta\cdot dt - kv^2 dt = \left\{g\sin\theta - kv^2\right\}dt$$

og

$$\frac{dv}{\left\{g\sin\theta - kv^2\right\}} = dt$$

höfum aðgreint breyturnar t og v og getum því heildað beint upphafsskilyrði

$v(t=0) = v(0) = 0$

$$\int_0^v \frac{dv'}{g\sin\theta - k(v')^2} = \int_0^t dt'$$

Hraði agnarinnar eykst

$$\rightarrow \frac{dv}{dt} > 0$$

því sést frá (*) að

$$g\sin\theta > kv^2$$

$$\frac{1}{\sqrt{kg\sin\theta}}\cdot \mathrm{Artanh}\left[\frac{\sqrt{k}\,v'}{\sqrt{g\sin\theta}}\right]\Bigg|_0^v = t$$

$$t = \frac{\mathrm{Artanh}\left\{\frac{\sqrt{k}\,v}{\sqrt{g\sin\theta}}\right\}}{\sqrt{kg\sin\theta}}$$

$$\mathrm{Artanh}\left[\frac{\sqrt{k}\,v}{\sqrt{g\sin\theta}}\right] = t\sqrt{kg\sin\theta}$$

$$\rightarrow \frac{\sqrt{k}\,v}{\sqrt{g\sin\theta}} = \tanh\left(t\sqrt{kg\sin\theta}\right)$$

og

$$v = \sqrt{\frac{g\sin\theta}{k}}\tanh\left(t\sqrt{kg\sin\theta}\right)$$

eða

$$\frac{dx}{dt} = \sqrt{\frac{g\sin\theta}{k}}\tanh\left(t\sqrt{kg\sin\theta}\right)$$

Þetta getum við heildað

$$\int_0^d dx = \int_0^t dt'\sqrt{\frac{g\sin\theta}{k}}\tanh\left(t'\sqrt{kg\sin\theta}\right)$$

$$d = \sqrt{\frac{g\sin\theta}{k}}\,\frac{\ln\left\{\cosh\left(t\sqrt{kg\sin\theta}\right)\right\}}{\sqrt{kg\sin\theta}}$$

$$\rightarrow d = \frac{1}{k}\ln\left\{\cosh\left(t\sqrt{kg\sin\theta}\right)\right\}$$

Henni er hægt að snúa við

$$\exp\{kd\} = \cosh\left(t\sqrt{kg\sin\theta}\right) \rightarrow$$

$$t\sqrt{kg\sin\theta} = \mathrm{ArCosh}\{\exp(kd)\}$$

eða

$$\boxed{t = \frac{\mathrm{ArCosh}\{\exp(kd)\}}{\sqrt{kg\sin\theta}}}$$

$$\lim_{kd\to 0} t = \frac{\sqrt{2\cdot kd}}{\sqrt{kg\sin\theta}} = \sqrt{\frac{2d}{g\sin\theta}}$$

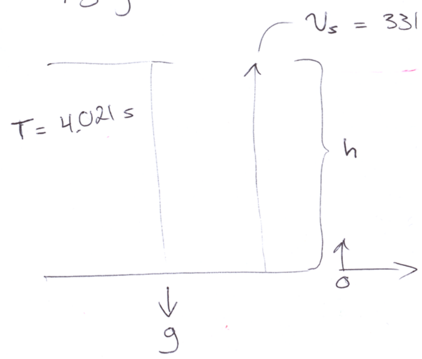
d fasti

$k \to 0$

þekkt lausn þegar enginn núningur er

2-30

Þyngdarhröðun



Lóðrétt fall $m\ddot{y} = -mg$

lausn

$$y = y_0 + \dot{y}_0 t - \frac{1}{2}gt^2$$

$$\left.\begin{array}{l} y(0) = h \\ \dot{y}(0) = 0 \end{array}\right\}\quad 0 = h - \frac{1}{2}gt$$

$$\rightarrow t_f = \sqrt{\frac{2h}{g}}$$

Hljóð upp $t_s = \frac{h}{v_s}$ $\rightarrow$ heildartími þ.t. hljóð heyrist er

$$T = t_s + t_f = \frac{h}{v_s} + \sqrt{\frac{2h}{g}}$$

þurfum að leysa

gefið

$$\boxed{\frac{h}{v_s} + \sqrt{\frac{2h}{g}} - T = 0}$$

Annarsstigs jafna fyrir breytuna $\sqrt{h}$

(9)

$$\sqrt{h} = \frac{-\sqrt{\frac{2}{g}} \pm \sqrt{\frac{2}{g} + \frac{4T}{v_s}}}{2/v_s} = \frac{v_s}{\sqrt{2g}}\left[-1 \pm \sqrt{1 + \frac{2gT}{v_s}}\right]$$

h er jákvæð stærð → tökum jákvæðu rótina

$\rightarrow h = 71.063$ m ef $g = 9.81\ m/s^2$

(10)

(2-40)

Ögn á braut

$$x(t) = A(2\alpha t - \sin(\alpha t))$$
$$y(t) = A(1 - \cos(\alpha t))$$

Hraðinn er alltaf snertill ~~við~~ brautina

Skilgreinum $\bar{v}(t) = v(t)\bar{T}(t)$ snertill brautar

$$\rightarrow \bar{a}(t) = \frac{d\bar{v}}{dt} = \frac{dv}{dt}\cdot\bar{T} + v\frac{d\bar{T}}{dt}$$

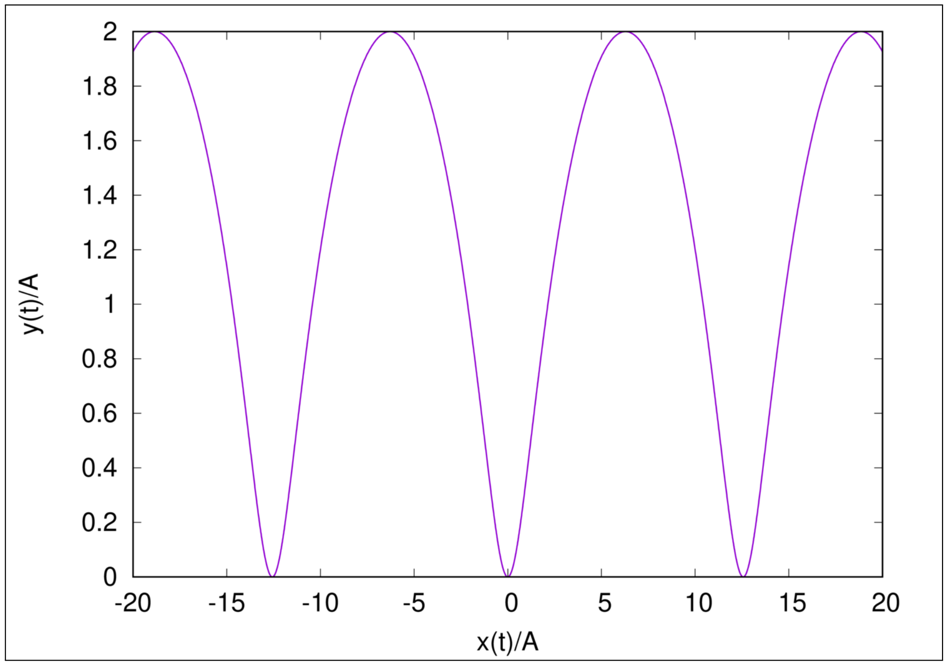
sjá mynd á næstu síðu

$$= a_t\bar{T} + a_n\bar{N}$$

$$\bar{T}\cdot\bar{T} = 1 \rightarrow \frac{d}{dt}(\bar{T}\cdot\bar{T}) = 0 = \bar{T}\cdot\frac{d\bar{T}}{dt} + \frac{d\bar{T}}{dt}\cdot\bar{T}$$
$$= 2\bar{T}\cdot\bar{N}$$

$\bar{N}$ og $\bar{T}$ eru hornréttir

Einingavigrar $\quad \bar{T}(t) = \frac{\dot{\bar{r}}(t)}{|\dot{\bar{r}}(t)|}, \quad \bar{N} = \frac{\dot{\bar{T}}(t)}{|\dot{\bar{T}}(t)|}$



(12)

$$\dot{x}(t) = A(2\alpha - \alpha\cos(\alpha t)) = A\alpha(2 - \cos(\alpha t))$$
$$\dot{y}(t) = A\alpha\sin(\alpha t)$$
$$\rightarrow v = \sqrt{\dot{x}^2 + \dot{y}^2} = A\alpha\sqrt{((2-\cos(\alpha t))^2 + \sin^2(\alpha t))}$$
$$= A\alpha\sqrt{5 - 4\cos(\alpha t)}$$

$$\rightarrow a_t = \frac{dv}{dt} = \frac{2A\alpha^2\sin(\alpha t)}{\sqrt{5 - 4\cos(\alpha t)}}$$

Varúð $\quad \dot{\bar{T}} \neq \bar{N} \rightarrow a_n \neq v \quad$ ($\dot{\bar{T}}$ er ekki stöðluður)

en $a^2 = a_t^2 + a_n^2$, og

$$a = \sqrt{(\ddot{x}(t))^2 + (\ddot{y}(t))^2} = A\alpha^2$$

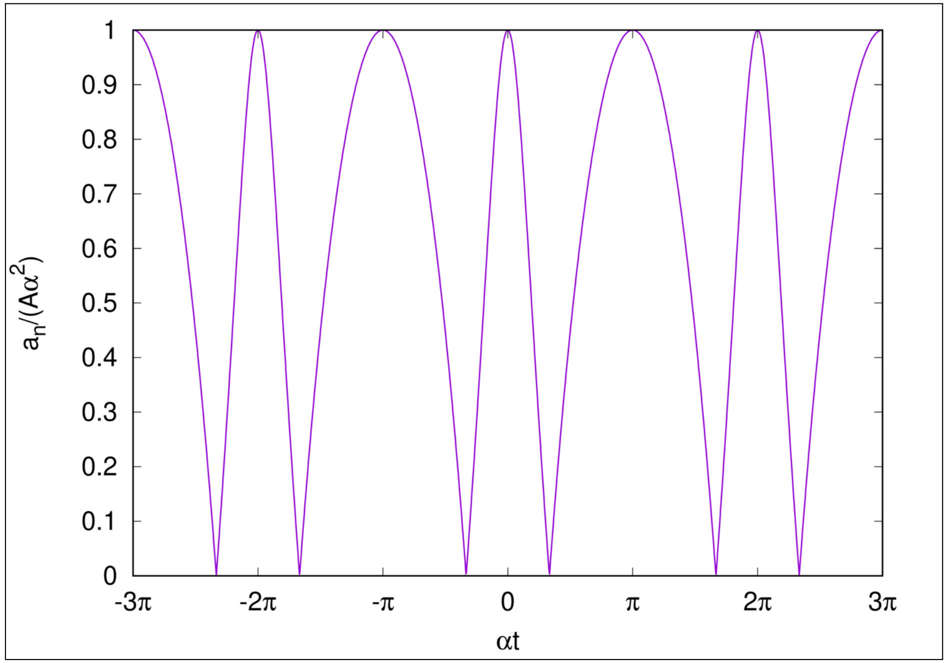
$$\rightarrow a_n = \sqrt{a^2 - a_t^2}$$

$$a_n = \sqrt{A^2\alpha^4 - \frac{4A^2\alpha^4\sin^2(\alpha t)}{5-4\cos(\alpha t)}} = A\alpha^2\sqrt{1-\frac{4\sin^2(\alpha t)}{5-4\cos(\alpha t)}}$$

$$= A\alpha^2\sqrt{\frac{5-4\cos(\alpha t)-4\sin^2(\alpha t)}{5-4\cos(\alpha t)}} = A\alpha^2\sqrt{\frac{1-4\cos(\alpha t)+4\cos^2(\alpha t)}{5-4\cos(\alpha t)}}$$

$$= A\alpha^2\sqrt{\frac{(1-2\cos(\alpha t))^2}{5-4\cos(\alpha t)}} = A\alpha^2\frac{|1-2\cos(\alpha t)|}{\sqrt{5-4\cos(\alpha t)}}$$

sjá graf á næstu síðu, sem sýnir max-gildi a_n í punktunum $\alpha t = n\pi$, $n \in \mathbb{Z}$



2-52

$$U(x) = U_0\left\{2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4\right\}$$ (sjá mynd á næstu síðu)

$U_0, a > 0$

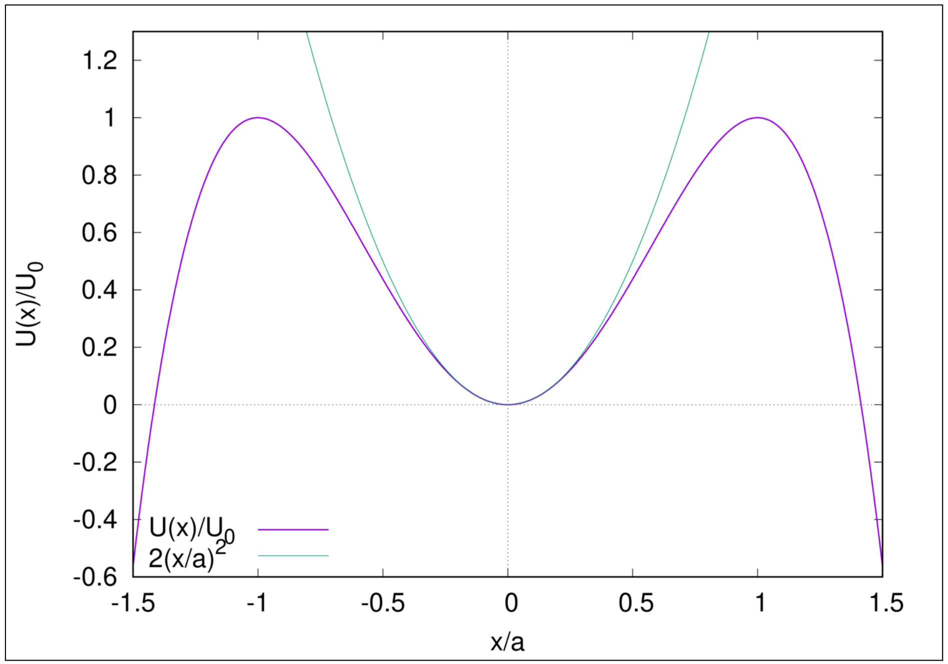
a) Krafturinn vegna mættisins $U(x)$

$$F(x) = -\frac{d}{dx}U(x) = -\frac{U_0}{a}\left\{4\left(\frac{x}{a}\right) - 4\left(\frac{x}{a}\right)^3\right\}$$

$$= -4\left(\frac{U_0}{a}\right)\left(\frac{x}{a}\right)\left\{1-\left(\frac{x}{a}\right)^2\right\}$$

b) stöðugt staðbundið lágmark þ. $x = 0$ því þar er $\frac{\partial^2 U}{\partial x^2} > 0$

óstöðug hámörk þegar $\left(\frac{x}{a}\right) = \pm$ því þar gildir að $\frac{\partial^2 U}{\partial x^2} < 0$



(17)

c) hver er horntíðni sveiflna ognær um $x = 0$

þar er mættið
$$U(x) \approx U_0 \frac{2}{a^2} x^2$$

Einvíður hreintóna sveifill er með hreyfijöfnu,
$$\ddot{x} + \omega^2 x = 0, \qquad \omega = \sqrt{\frac{k}{m}}$$

mættið $U(x) = \frac{1}{2}kx^2$ og kraftinn $F(x) = -kx$

því er horntíðnin hér
$$\omega = \sqrt{\frac{4U_0}{ma^2}}$$

d) minnsti hraði fyrir ögn til að sleppa frá lágmarki í $x=0$. Þá verði hreyfiorkan í $x=0$ jöfn mættisorku í $x = \pm a$
$$\frac{m v_m^2}{2} = U(a) = U_0 \;\rightarrow\; v_{min} = \sqrt{\frac{2U_0}{m}}$$

(18)

e) Ef í $t=0$ $x(0) = 0$ og $v = v_{min}$

Hér er textinn ekki skýr, en ég geri ráð fyrir að höfundur vilji vita hvernig þessi markhreyfing lítur út frá $x=0$ að $x=a$

Orkan er varðveitt, í $x=0$ er
$$E_{min} = \frac{m v_{min}^2}{2}, \qquad E_{min} = U_0$$

og í hverjum punkti
$$E_{min} = \frac{mv^2}{2} + U(x) = \frac{mv^2}{2} + U_0\left\{2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4\right\}$$
$$\rightarrow U_0 = \frac{mv^2}{2} + U_0\left\{2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4\right\}$$
$$\rightarrow \frac{mv^2}{2} = U_0 - U_0\left\{2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4\right\} = U_0\left\{1 - 2\left(\frac{x}{a}\right)^2 + \left(\frac{x}{a}\right)^4\right\}$$

(19)

$$v = \sqrt{\frac{2U_0}{m}\left\{1 - 2\left(\frac{x}{a}\right)^2 + \left(\frac{x}{a}\right)^4\right\}} = v(x)$$

en $v = \frac{dx}{dt} \rightarrow dt = \frac{dx}{v(x)}$
$$\rightarrow t = \sqrt{\frac{m}{2U_0}} \int_0^x \frac{dx'}{\sqrt{1 - 2\left(\frac{x'}{a}\right)^2 + \left(\frac{x'}{a}\right)^4}} = \sqrt{\frac{ma^2}{2U_0}} \int_0^{\frac{x}{a}} \frac{du}{\sqrt{1 - 2u^2 + u^4}}$$

þegar $\frac{x}{a} < 1$
$$t = \sqrt{\frac{ma^2}{8U_0}}\left\{\ln \frac{1 + \frac{x}{a}}{1 - \frac{x}{a}}\right\} \rightarrow \frac{t}{\sqrt{\frac{ma^2}{8U_0}}} = \ln\left(\frac{a+x}{a-x}\right)$$

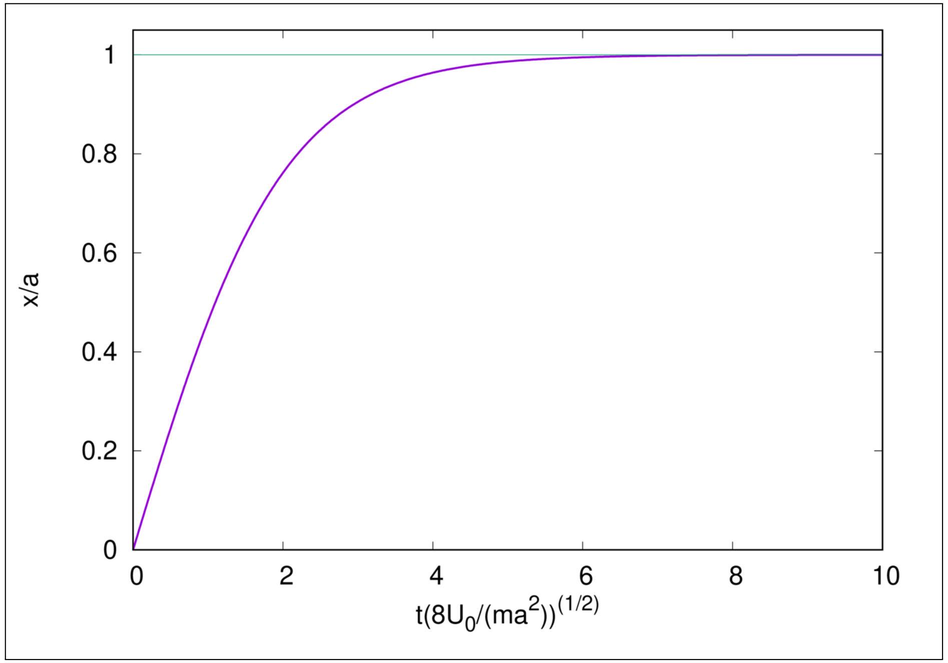
(20)

og því
$$\exp\left\{t\sqrt{\frac{8U_0}{ma^2}}\right\} = \frac{a+x}{a-x}$$
$$\rightarrow x(t) = a\,\frac{\left\{\exp\left(t\sqrt{\frac{8U_0}{ma^2}}\right) - 1\right\}}{\left\{\exp\left(t\sqrt{\frac{8U_0}{ma^2}}\right) + 1\right\}}$$

með $\lim_{t \to \infty} x(t) = a$

ögnin tekur óendanlegan tíma að komast í $x = a$

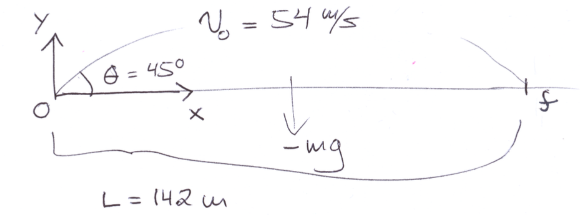
sjá mynd á næstu síðu



(2-55) 22

Grasker $m = 5\,\text{kg}$, $F = -kmv$

ákvarða k



Hreyfijöfnur

x-þáttur: $ma_x = F_x = -mkv_x$

eða $m\frac{dv_x}{dt} = -mkv_x \to \boxed{\frac{dv_x}{v_x} = -k\,dt}$ (1)

eins er hægt að fá

$\frac{dv_x}{dt} = -k\frac{dx}{dt} \to \boxed{dv_x = -k\,dx}$ (2)

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Heildum (2)

$$\int_{v_{x0}}^{v_{xf}} dv_x' = -k\int_0^{x_f} dx'$$

$$v_{xf} - v_{x0} = -kx_f$$

Heildum (1)

$$\int_{v_{x0}}^{v_{xf}} \frac{dv_x'}{v_x'} = -k\int_0^{t_f} dt$$

$$\ln\left\{\frac{v_{xf}}{v_{x0}}\right\} = -kt_f$$

$$v_{xf} = v_{x0}\exp(-kt_f)$$

$$\boxed{x_f = \frac{v_{x0}}{k}\left[1 - \exp(-kt_f)\right]} \quad (*)$$

y-þáttur:

$$m\frac{dv_y}{dt} = -mg - mkv_y$$

$$\to \frac{dv_y}{dt} = -g - kv_y$$

$$\to \boxed{\frac{dv_y}{g + kv_y} = -dt} \quad (3)$$

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Eins má umrita hreyfijöfnuna sem

$$\frac{dv_y}{dt} = -g - k\frac{dy}{dt} \to \frac{dv_y}{dt} + \frac{k\,dy}{dt} = -g$$

eða $dv_y + k\,dy = -g\,dt$

notum $\frac{dy}{dt} = v_y \to dt = \frac{dy}{v_y}$

$$dv_y + k\,dy = -g\frac{dy}{v_y} \to dv_y = -k\,dy - g\frac{dy}{v_y}$$

$$= -\left(k + \frac{g}{v_y}\right)dy$$

$$\to \frac{dy}{v_y} = -\frac{dv_y}{(kv_y + g)}$$

eða

$$\boxed{dy = -\frac{v_y\,dv_y}{(kv_y + g)}} \quad (4)$$

Heildum (3)

$$\int_{v_{y0}}^{v_{yf}} \frac{dv_y}{g+kv_y} = -\int_0^{t_f} dt$$

sem gefur

$$\frac{1}{k}\ln\left(\frac{g+kv_{yf}}{g+kv_{y0}}\right) = -t_f$$

eða

$$\boxed{g+kv_{yf} = (g+kv_{y0})\exp(-kt_f)} \quad (5)$$

heildum (4)

$$\int_0^0 dy = -\int_{v_{y0}}^{v_{yf}} \frac{v_y\,dv_y}{(kv_y+g)}$$

$$\boxed{0 = \frac{g}{k^2}\ln\left\{\frac{kv_{yf}+g}{kv_{y0}+g}\right\} - \left(\frac{v_{yf}-v_{y0}}{k}\right)} \quad (6)$$

Notum (5) og (6) til að losna við v_{yf}, sem er óþekkt stærð.

$$\frac{g}{k^2}\ln\left(\frac{kv_{yf}+g}{kv_{y0}+g}\right) = -\frac{g}{k}t_f$$

$$0 = -\frac{g}{k}t_f - \frac{v_{yf}-v_{y0}}{k}$$

$$\frac{g}{k}t_f = -\frac{v_{yf}}{k} + \frac{v_{y0}}{k}$$

$$\frac{g}{k}t_f = +\frac{v_{y0}}{k} - \frac{1}{k^2}(g+kv_{y0})e^{-kt_f} + \frac{g}{k^2}$$

$$= +\frac{1}{k^2}(g+kv_{y0})(1-e^{-kt_f})$$

$$\to \boxed{(1-e^{-kt_f}) = \frac{gkt_f}{g+kv_{y0}}} \quad (7)$$

tvær jöfnur, tvær óþekktar stærðir t_f og k

leysum saman með (*):

$$\boxed{x_f = \frac{v_{x0}}{k}\left[1-e^{-kt_f}\right]} \quad (8)$$

(x_f, v_{0x}, v_{0y} þekkt)

ólínulegar

$$e^{-kt_f} = 1 - \frac{x_f k}{v_{x0}}$$

$$-kt_f = \ln\left(1-\frac{x_f k}{v_{x0}}\right) \quad (9)$$

sem líka hefur fengið

(7) og (8)

$$\frac{x_f k}{v_{x0}} = \frac{gkt_f}{g+kv_{y0}}$$

(9)

$$\frac{x_f k}{v_{x0}} = -\frac{g}{g+kv_{y0}}\ln\left(1-\frac{x_f k}{v_{x0}}\right)$$

ein jafna, ein óþekkt stærð, k

$$-\frac{x_f k}{v_{x0}}\frac{(g+kv_{y0})}{g} = \ln\left(1-\frac{x_f k}{v_{x0}}\right)$$

þ $k \to 0$ stefnir hægri hliðin einnig á x_f, eins og ætti að vera

$$\to \exp\left\{-\frac{x_f k(g+kv_{y0})}{v_{x0}\cdot g}\right\} = 1-\frac{x_f k}{v_{x0}}$$

$$\boxed{x_f = \frac{v_{x0}}{k}\left\{1-\exp\left[-\frac{x_f k(g+kv_{y0})}{v_{x0}\cdot g}\right]\right\}}$$

ólínuleg jafna sem við leysum tölulega

$v_0 = 54$ m/s, $x_f = 142$ m, $v_{0y} = 54\cdot\cos\left(\frac{\pi}{4}\right) \approx 38.2$ m/s

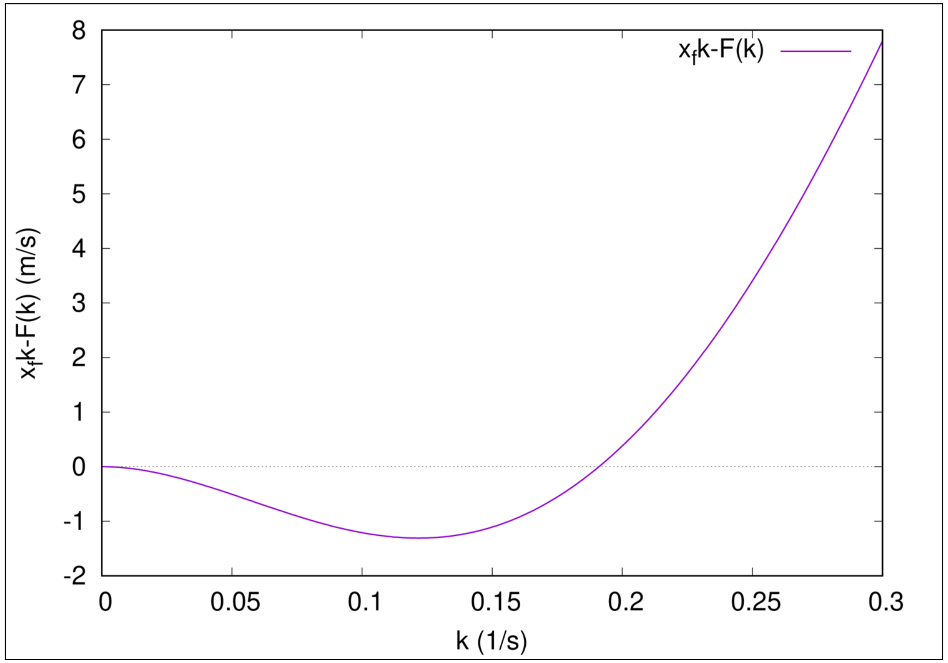
$\theta = 45^\circ$, $g = 9.81$ m/s^2

$x_f k = F(k)$

leysum

$$x_f k - F(k) = 0$$

Grafið á næstu síðu sýnir að $k=0$ er lausn, eins og við höfðum séð áður, en líka sést önnur lausn sem wxMaxima gefur sem $k = 0{,}19105\ldots\ s^{-1}$



(%i2) find_root(142·x−38.2·(1−exp(−(142·x·(9.81+x·38.2))/(38.2·9.81)))=0, x, 0.1, 0.3);

(%o2) 0.1910528801334695

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