① Dæmi 12-03 í bók

Kerfi tveggja tengdra hreintóna sveifla, lýst með

$$\begin{cases} \ddot{x}_1 + \frac{m}{M}\ddot{x}_2 + \omega_0^2 x_1 = 0 \\ \ddot{x}_2 + \frac{m}{M}\ddot{x}_1 + \omega_0^2 x_2 = 0 \end{cases}$$

Setjum

$$\bar{x}(t) = \begin{pmatrix} B_1 e^{i\omega t} \\ B_2 e^{i\omega t} \end{pmatrix} \to \ddot{\bar{x}}(t) = -\omega^2 \begin{pmatrix} B_1 e^{i\omega t} \\ B_2 e^{i\omega t} \end{pmatrix}$$

$$= -\omega^2 \bar{x}(t)$$

→ Umskrifum þetta sem hneppi

$$\begin{pmatrix} 1 & \frac{m}{M} \\ \frac{m}{M} & 1 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \omega_0^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

því fæst

$$\begin{pmatrix} 1 & \frac{m}{M} \\ \frac{m}{M} & 1 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \frac{\omega_0^2}{\omega^2}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

Sem er greinilega eigingildisverkefni

Sem hefur eigingildi

$$\frac{\omega_0^2}{\omega^2} = 1 - \frac{m}{M} \quad \text{með eiginvigur} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \omega = \frac{\omega_0}{\sqrt{1-\frac{m}{M}}}$$

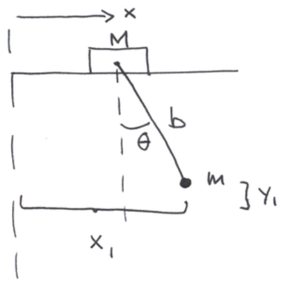
$$\frac{\omega_0^2}{\omega^2} = 1 + \frac{m}{M} \quad \text{með eiginvigur} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \omega = \frac{\omega_0}{\sqrt{1+\frac{m}{M}}}$$

Andsamhverfur

samhverfur sveifluháttur

Lægri tíðni

② Dæmi 12-18 í bók



Notum kartísk hnit til að raða saman

$$x_1 = x + b\sin\theta \qquad y_1 = b - b\cos\theta$$

$$\downarrow \qquad\qquad \downarrow$$

$$\dot{x}_1 = \dot{x} + \dot{\theta} b \cos\theta \qquad \dot{y}_1 = \dot{\theta} b \sin\theta$$

$$\to T = \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left\{\dot{x}_1^2 + \dot{y}_1^2\right\}$$

$$= \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left\{(\dot{x} + \dot{\theta} b\cos\theta)^2 + (\dot{\theta} b \sin\theta)^2\right\}$$

$$= \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left\{\dot{x}^2 + (b\dot{\theta})^2 + 2b\dot{x}\dot{\theta}\cos\theta\right\}$$

$$U = mgy_1 = mgb\{1 - \cos\theta\} \simeq mgb\left\{1 - 1 + \frac{\theta^2}{2} + \cdots\right\}$$

$$= \frac{mgb}{2}\theta^2$$

Tökum saman liði í T

$$T = \left(\frac{M+m}{2}\right)\dot{x}^2 + \frac{m}{2}\left((b\dot{\theta})^2 + 2b\dot{x}\dot{\theta}\right)$$

fyrir litla horn $\cos\theta \sim 1 + \cdots$

$$M = \begin{pmatrix} \frac{\partial^2 T}{\partial \dot{x}^2} & \frac{\partial^2 T}{\partial \dot{x}\partial\dot{\theta}} \\ \frac{\partial^2 T}{\partial\dot{\theta}\partial\dot{x}} & \frac{\partial^2 T}{\partial\dot{\theta}^2} \end{pmatrix} = \begin{pmatrix} M+m & mb \\ mb & mb^2 \end{pmatrix}$$

ekki sama vídd

$$A = \begin{pmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x\partial\theta} \\ \frac{\partial^2 U}{\partial x\partial\theta} & \frac{\partial^2 U}{\partial\theta^2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & mgb \end{pmatrix}$$

$A\bar{a} = \omega^2 M\bar{a}$ hér tapast samhverfa ef reynt er að margfalda með M^{-1}

lítum því á jöfnuhneppið

$$(A-\omega^2 M)\bar{a} = 0 \quad \longleftrightarrow \quad \begin{pmatrix} -\omega^2(M+m) & -\omega^2 mb \\ -\omega^2 mb & mgb-\omega^2 mb^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

Svo lausn sé til verður ákveðan að hverfa

$$\omega^2(M+m)\{\omega^2 mb^2 - mgb\} - \omega^4 m^2 b^2 = 0$$

$$\left.\omega^2\{\omega^2 Mb^2 m - mgb(m+M)\} = 0 \right\} \longrightarrow \quad \omega_1 = 0, \quad \omega_2 = \sqrt{\frac{g}{Mb}(M+m)}$$

Eiginvigur ω_1 verður að vera $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Fyrir ω_2:

$$\begin{pmatrix} -\omega_2^2(M+m) & -\omega_2^2 mb \\ -\omega_2^2 mb & mgb-\omega_2^2 mb^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$= \omega_2^2\begin{pmatrix} -(M+m) & -mb \\ -mb & -\frac{m^2b^2}{M+m} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \quad \longrightarrow \quad \begin{pmatrix} -\frac{bm}{M+m} \\ 1 \end{pmatrix}$$

eiginvigur

$$\longrightarrow \quad a_1 = -\frac{bm}{M+m}a_2$$

$$U = \begin{pmatrix} 1 & -\frac{bm}{M+m} \\ 0 & 1 \end{pmatrix}$$

$$\bar{a} = \begin{pmatrix} x \\ \theta \end{pmatrix} = U\bar{\eta} = \begin{pmatrix} \eta_1 - \eta_2\frac{bm}{M+m} \\ \eta_2 \end{pmatrix}$$

$$\left.\begin{aligned} \longrightarrow \quad & \eta_2 \sim \theta \\ & \eta_1 - \eta_2\frac{bm}{M+m} \sim x \end{aligned}\right\} \longrightarrow \quad \eta_1 \sim x + \frac{bm}{M+m}\theta$$

Sveifluháttur η_2 verður þegar $\eta_1 = 0$ eða $x = -\frac{bm}{M+m}\theta$

η_1 verður þegar $\eta_2 = 0$ eða $\theta = 0$

3 Dæmi 12-23 í bók

Sýna að orka eiginsveifluhátts sé fasti

sýna fyrir síðan fyrir 12.03

Eiginsveifluháttur $\eta_r = \beta_r e^{i\omega_r t} \longrightarrow \dot{\eta}_r = i\omega_r\beta_r e^{i\omega_r t} = i\omega_r\eta_r$

Orkan

$$E_r = T_r + U_r = \frac{1}{2}\dot{\eta}_r^2 + \frac{1}{2}\omega_r^2\eta_r^2$$

Annarsstig, eða annað veldið af η_r þarf alltaf að taka með varúð, Hér er heppilegt að nota raunhlutann $\eta_r^2 \longrightarrow \{\mathrm{Re}\,\eta_r\}^2$

þ.a.

$$E_r = \frac{1}{2}(\mathrm{Re}\,\dot{\eta}_r)^2 + \frac{1}{2}\omega_r^2(\mathrm{Re}\,\eta_r)^2$$

$$(\mathrm{Re}\,\eta_r)^2 = \left\{\mathrm{Re}\left[(\beta_r' + i\beta_r'')(\cos(\omega_r t) + i\sin(\omega_r t))\right]\right\}^2$$

$$= \left\{\beta_r'\cos(\omega_r t) - \beta_r''\sin(\omega_r t)\right\}^2$$

$$(\mathrm{Re}\,\dot{\eta}_r)^2 = \left\{\mathrm{Re}\left[i\omega_r(\beta_r' + i\beta_r'')(\cos(\omega_r t) + i\sin(\omega_r t))\right]\right\}^2$$

$$= \left\{-\omega_r(\beta_r''\cos(\omega_r t) + \beta_r'\sin(\omega_r t))\right\}^2$$

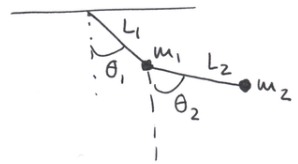
$$\to E_r + T_r + U_r = \frac{1}{2}\omega_r^2\left\{(\beta_r')^2 + (\beta_r'')^2\right\}$$

$$= \frac{1}{2}\omega_r^2|\beta_r|^2$$

Ég lét ~~það~~ nægja, því sveifluhættirnir eru hornréttir → orka flyst ekki milli þeirra. Samanber t.d. eiginástand jöfnu Schrödingers í skammtafræði . . .

④ Dæmi 12-27 í bók

Tvöfaldur sveifill



Við höfum leitt út áður

$$T = \frac{m_1}{2}(L_1\dot{\theta}_1)^2 + \frac{m_2}{2}\left\{(L_1\dot{\theta}_1)^2 + (L_2\dot{\theta}_2)^2 - 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right\}$$

$$\approx \frac{\dot{\theta}_1^2}{2}\left\{m_1L_1^2 + m_2L_2^2\right\} + \frac{m_2}{2}(L_2\dot{\theta}_2)^2 - m_2L_1L_2\dot{\theta}_1\dot{\theta}_2$$

$$\to M = \begin{pmatrix} (m_1+m_2)L_1^2 & -m_2L_1L_2 \\ -m_2L_1L_2 & m_2L_2^2 \end{pmatrix}$$

$$U = m_1gL_1\{1 - \cos\theta_1\} + m_2g\{L_1(1-\cos\theta_1) + L_2(1-\cos\theta_2)\}$$

$$\approx \{m_1 + m_2\}gL_1\frac{\theta_1^2}{2} + m_2gL_2\frac{\theta_2^2}{2}$$

og því

$$A = \begin{pmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2gL_2 \end{pmatrix}$$

$$(A - \omega^2 M)\bar{a} = 0$$

$$\begin{pmatrix} (m_1+m_2)gL_1 - \omega^2(m_1+m_2)L_1^2 & +\omega^2 m_2L_1L_2 \\ +\omega^2 m_2L_1L_2 & m_2gL_2 - \omega^2 m_2L_2^2 \end{pmatrix}\bar{a} = 0$$

Skrifum sem

$$\begin{pmatrix} A-\omega^2 B & \omega^2 x \\ \omega^2 x & C-\omega^2 D \end{pmatrix} \rightarrow \det\left(A-\omega^2 M\right)$$

$$= (A-\omega^2B)(C-\omega^2D) - \omega^4 x^2 = 0$$

$$\rightarrow \omega^4(x^2-BD) + \omega^2(BC+AD) - AC = 0$$

$$\rightarrow \omega^2 = \frac{-BC+AD \pm \sqrt{(BC+AD)^2 + 4(x^2-BD)AC}}{2(x^2-BD)}$$

$$x^2 - BD = (m_2L_1L_2)^2 - (m_1+m_2)L_1^2 m_2 L_2^2 = -m_1m_2L_1^2L_2^2$$

$$BC+AD = (m_1+m_2)L_1^2 m_2 g L_2 + (m_1+m_2) g L_1 m_2 L_2^2$$

$$= (m_1+m_2)m_2 g\left(L_1^2L_2 + L_1L_2^2\right)$$

athugið víddina

$$\rightarrow \omega_\pm^2 = \frac{(m_1+m_2)g(L_1+L_2) \pm \sqrt{(m_1+m_2)g^2\left\{m_1(L_1-L_2)^2 + m_2(L_1+L_2)^2\right\}}}{2m_1L_1L_2}$$

$$\begin{pmatrix} A-\omega_\pm^2 B & \omega_\pm^2 x \\ \omega_\pm^2 x & C-\omega_\pm^2 D \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\left.\begin{aligned} (A-\omega_\pm^2B)a + \omega_\pm^2 x b &= 0 \\ \omega_\pm^2 x a + (C-\omega_\pm^2D)b &= 0 \end{aligned}\right\} \rightarrow \left.\frac{a}{b}\right|_\pm = -\frac{\omega_\pm x}{(A-\omega_\pm^2 B)}$$

Hér er hægt að sýna að $(A-\omega_+^2B) < 0$ og $(A-\omega_-^2B) > 0$

því er hægt að sýna að

jákvæð stærð

$$\eta_+ \sim \theta_1 - \cdots\theta_2$$

$$\eta_- \sim \theta_1 + \cdots\theta_2$$

$\rightarrow$ η_+ er  andfasi

η_- er  í fasa

(5) Dæmi 12-08 í bók

Notum sama dæmi aftur, en einfaldara því $m_1 = m_2$, $L_1 = L_2$

köllum m og l

$$\rightarrow A = mgl\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad M = ml^2\begin{pmatrix} 2 & +1 \\ +1 & 1 \end{pmatrix}$$

$$\det\left(A-\omega^2M\right) = \begin{vmatrix} 2mgl - 2ml^2\omega^2 & -\omega^2ml^2 \\ -\omega^2 ml^2 & mgl - ml^2\omega^2 \end{vmatrix}$$

$$\rightarrow \begin{vmatrix} \frac{2g}{l} - 2\omega^2 & -\omega^2 \\ -\omega^2 & \frac{g}{l} - \omega^2 \end{vmatrix} = 0$$

$$\rightarrow \left(\frac{2g}{l} - 2\omega^2\right)\left(\frac{g}{l} - \omega^2\right) - \omega^4 = 0$$

7-07 í bók

$$T = \frac{ml^2}{2}\left(2\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2\right)$$

$$U = \frac{mgl}{2}\left(2\phi_1^2 + \phi_2^2\right)$$

$$\rightarrow 2\left(\frac{g}{l}\right)^2 - \omega^2\left(\frac{2g}{l} + \frac{2g}{l}\right) + 2\omega^4 - \omega^4 = 0$$

$$\rightarrow \omega^4 - 4\frac{g}{l}\omega^2 + 2\left(\frac{g}{l}\right)^2 = 0$$

$$\rightarrow \omega^2 = \frac{4\frac{g}{l} \pm \sqrt{\left(16\left(\frac{g}{l}\right)^2 - 8\left(\frac{g}{l}\right)^2\right)}}{2} = \left\{2 \pm \sqrt{2}\right\}\frac{g}{l}$$

$$\rightarrow \omega_1 = \sqrt{2+\sqrt{2}}\sqrt{\frac{g}{l}}$$

$$\omega_2 = \sqrt{2-\sqrt{2}}\sqrt{\frac{g}{l}}$$

þurfum að leysa $\left(A - \omega_{1,2}^2 M\right)\bar{a} = 0$

(17)

Hér fæst fyrir ω_1

$$a_2 = -\sqrt{2}\, a_1$$

og fyrir ω_2

$$a_2 = \sqrt{2}\, a_1$$

eiginvigrar $\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$: ω_1 $\qquad \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$: ω_2

$$\rightarrow U = \begin{pmatrix} 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{pmatrix} \qquad U^t = \begin{pmatrix} 1 & -\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix} \qquad \eta = U^t \bar{a} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

↑•↓ ↓•↑

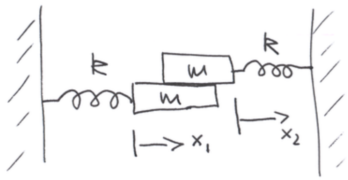
$\rightarrow \eta_1 \sim x_1 - \sqrt{2}\, x_2$ $\qquad$ η_1 gerist þ. $\eta_2 = 0$ þ.e. $x_1 = \sqrt{2}\, x_2$

$\eta_2 \sim x_1 + \sqrt{2}\, x_2$ $\qquad$ η_2 gerist þ. $\eta_1 = 0$ þ.e. $x_1 \sim -\sqrt{2}\, x_2$

↑•↓ ↑•↓

(18)

6) Dæmi 12-06 í bók



núningskraftur á 1 vegna 2

$$f = -\beta(\dot{x}_1 - \dot{x}_2)$$

$$\rightarrow \begin{cases} m\ddot{x}_1 + \beta(\dot{x}_1 - \dot{x}_2) + kx_1 = 0 \\ m\ddot{x}_2 + \beta(\dot{x}_2 - \dot{x}_1) + kx_2 = 0 \end{cases}$$

ekki geymið kerfi

Reynum lausnir $\quad x_1(t) = A e^{\alpha t} \qquad x_2(t) = B e^{\alpha t}, \quad \alpha \in \mathbb{C}$

$$m\alpha^2 A + \beta\alpha(A - B) + kA = 0$$
$$m\alpha^2 B + \beta\alpha(B - A) + kB = 0$$

(19)

~~þá~~

$$\begin{pmatrix} m\alpha^2 + k + \beta\alpha & -\beta\alpha \\ -\beta\alpha & m\alpha^2 + k + \beta\alpha \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

ákveðan

$$\rightarrow (m\alpha^2 + \beta\alpha + k)^2 - (\beta\alpha)^2 = 0$$

$$\rightarrow m\alpha^2 + \beta\alpha + k = \pm\beta\alpha \quad \rightarrow m\alpha^2 + k = 0$$

eða $\quad m\alpha^2 + 2\beta\alpha + k = 0$

$$\alpha^2 = -\frac{k}{m} \longrightarrow \alpha_1 = \pm i\sqrt{\frac{k}{m}}$$

$$\alpha^2 + 2\frac{\beta}{m}\alpha + \frac{k}{m} = 0 \rightarrow \alpha_2 = \frac{1}{m}\left\{-\beta \pm \sqrt{\beta^2 - km}\right\}$$

(20)

α_1 - hátturinn er sveiflulausn með $\omega_1 = \sqrt{\frac{k}{m}}$

Lausnin er því

$$x_1(t) = A_1^+ e^{i\omega_1 t} + A_1^- e^{-i\omega_1 t} + e^{-\frac{\beta}{m}t}\left\{B_1^+ e^{+\frac{\sqrt{\beta^2 - km}}{m}t} + B_1^- e^{-\frac{\sqrt{\beta^2 - km}}{m}t}\right\}$$

$$x_2(t) = \underbrace{A_2^+ e^{i\omega_1 t} + A_2^- e^{-i\omega_1 t}}_{\text{sveiflandi Eiginþáttur}} + e^{-\frac{\beta}{m}t}\underbrace{\left\{B_2^+ e^{+\frac{\sqrt{\beta^2 - km}}{m}t} + B_2^- e^{-\frac{\sqrt{\beta^2 - km}}{m}t}\right\}}_{\text{deyfður háttur}}$$

sveiflandi Eiginþáttur

→ engin innbyrðishreyfing

→ í fasa

deyfður háttur

→ í andfasa