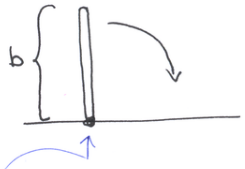
1 Dæmi 11-20 í bók

einsleit stöng með lengd fellur
finna hornferðina þ. hún skellur á gólfið.



Einfaldast að nota orkuvarðveislu

Í upphafi $E_1 = U = mg\frac{b}{2}$ ← m.v. CM

Í lok $E_2 = T = \frac{I}{2}\omega^2$

um endapunkt $I = I_{33} = \int_0^b dx\,\rho_l\left(x^2 - z^2\right) = \rho_l \int_0^b dx\cdot x^2$

$$= \rho_l \frac{1}{3} b^3 = \frac{1}{3}(\rho_l b)b^2 = \frac{1}{3}mb^2$$

$$\to E_2 = \frac{1}{6}mb^2\omega^2 = E_1 = mg\frac{b}{2} \to \omega = \sqrt{\frac{3g}{b}}$$

2 Dæmi 11-22 og -23 í bók

$$tr\{\mathbb{I}\} = \sum_k I_{kk}$$

sýnið að einslögunar ummyndun breyti ekki sporinu þess
Einóta ummyndun

$\mathbb{I}' = \lambda\mathbb{I}\lambda^t$ eins gildir $\lambda\lambda^t = \mathbb{1}$

$$tr\{\mathbb{I}'\} = tr\{\lambda\mathbb{I}\lambda^t\} = tr\{\lambda^t\lambda\mathbb{I}\} = tr\{\mathbb{I}\}$$

sýnið að samskonar gildi fyrir ákveðu

$$\det\{\lambda\mathbb{I}\lambda^t\} = \det\{\lambda\}\det\{\mathbb{I}\}\det\{\lambda^t\}$$

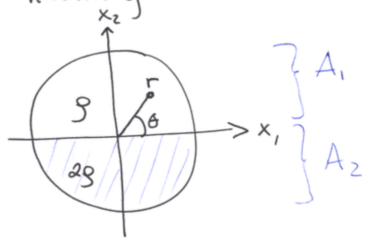
$$\det\{\lambda\lambda^t\} = \det\{\mathbb{1}\} = 1$$

$$\to \det\{\lambda\mathbb{I}\lambda^t\} = \det\{\mathbb{I}\}$$

Skoðið endilega
„The matrix Cookbook"
á veraldarvefnum

3 Dæmi 11-25 í bók

þunnur diskur settur saman úr tveimur einsleitum helmingum



finnum fall Lagrange fyrir veltu disksins.

Við þurfum að finna CM og I í gegnum CM

Miðað við teikninguna hér að framan er CM fyrir neðan O-hnitapunktinn á x_2-ás

$$x_2^{CM} = \frac{\rho}{M}\left\{2\int_{A_2} x_2\,dA + \int_{A_1} x_2\,dA\right\}$$

$$= \frac{\rho}{M}\left\{2\int_0^R r\,dr\int_{\pi}^{2\pi} d\theta\; r\sin\theta + \int_0^R r\,dr\int_0^{\pi} d\theta\; r\sin\theta\right\}$$

$$= \frac{\rho}{M}\frac{R^3}{3}\left\{-2\cos\theta\Big|_{\pi}^{2\pi} - \cos\theta\Big|_0^{\pi}\right\} = \frac{\rho R^3}{3M}\left\{-2(1+1)-(-1-1)\right\}$$

$$= -\frac{2\rho R^3}{3M}$$

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Massi diskins er

$$M = \frac{2\cdot\pi R^2 \varrho}{2} + \frac{\pi R^2 \varrho}{2} = \frac{3}{2}\varrho\pi R^2$$

$$\to \quad x_2^{CM} = -\frac{2\varrho R^3}{3}\,\frac{2}{3\varrho\pi R^2} = -\frac{4R}{9\pi}$$

Næst reiknum við I m.v. O, þ.e. miðju hjóls

$$I_{33} = \varrho\left\{2\int_0^R (r^2-z^2)\, r\,dr \int_\pi^{2\pi} d\theta + \int_0^R (r^2-z^2)\, r\,dr \int_0^\pi d\theta\right\}$$

$z=0$

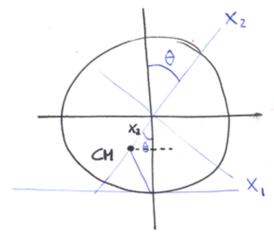
$$= \varrho\frac{R^4}{4}\left\{2\pi + \pi\right\} = \frac{3}{4}\varrho\pi R^4 = \frac{1}{2}MR^2 = I_3$$

6

Notum setningu Steiners til að reikna I_3 í gegnum CM

$$I_0 = I_3 - M(x_2^{CM})^2 = \frac{1}{2}MR^2 - M\frac{16R^2}{81\pi^2} = \frac{1}{2}MR^2\left[1 - \frac{32}{81\pi^2}\right]$$

Þurfum hraða CM



Í föstu hnitakerfi breytur

$$x_{CM} = R\theta - |x_2^{CM}|\sin\theta$$
$$y_{CM} = R - |x_2^{CM}|\cos\theta$$

$$\dot{x}_{CM} = R\dot\theta - |x_2^{CM}|\dot\theta\cos\theta$$
$$\dot{y}_{CM} = |x_2^{CM}|\dot\theta\sin\theta$$

$$V^2 = (\dot{x}_{CM}^2 + \dot{y}_{CM}^2) = \left\{R\dot\theta - |x_2^{CM}|\dot\theta\cos\theta\right\}^2 + \left\{|x_2^{CM}|\dot\theta\sin\theta\right\}^2$$

$$= (R\dot\theta)^2 + (|x_2^{CM}|\dot\theta)^2 - 2\dot\theta^2 R|x_2^{CM}|\cos\theta$$

$$= \dot\theta^2\left\{R^2 + |x_2^{CM}|^2 - 2R|x_2^{CM}|\cos\theta\right\}$$

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$$V^2 = \dot\theta^2 R^2\left\{1 + \frac{16}{81\pi^2} - \frac{8}{9\pi}\cos\theta\right\}$$

$$T = \frac{1}{2}MV^2 + \frac{I_0}{2}\dot\theta^2 = \frac{1}{2}M(R\dot\theta)^2\left\{1 + \frac{16}{81\pi^2} - \frac{8}{9\pi}\cos\theta\right\}$$
$$+ \frac{1}{2}M(R\dot\theta)^2\left\{\frac{1}{2} - \frac{16}{81\pi^2}\right\}$$

$$= \frac{1}{2}M(R\dot\theta)^2\left\{\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right\}$$

$$U = Mg\,y_{CM}, \qquad y_{CM} = R - |x_2^{CM}|\cos\theta = R\left\{1 - \frac{4}{9\pi}\cos\theta\right\}$$

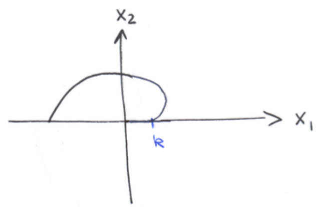
Við getum notað $U = Mg\,y_{CM} - Mg\frac{R}{2} = \frac{1}{2}Mg\left\{\frac{R}{2} - \frac{8}{9\pi}\cos\theta\right\}$

8

$$\to L = \frac{1}{2}M(R\dot\theta)^2\left\{\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right\} - \frac{1}{2}Mg\left\{\frac{R}{2} - \frac{8}{9\pi}\cos\theta\right\}$$

4 Dæmi 11-19 í bók

Plata (einsleit) með ϱ takmörkuð af $\boxed{r = ke^{\alpha\theta}}$ og $\theta = 0$ og $\theta = \pi$. Finnum hverfitregðuþininn um O $(r=0)$ ef platan er í x_1-x_2-sléttunni



notum pól hnit

$$x_1 = r\cos\theta$$
$$x_2 = r\sin\theta$$

$$I_{11} = \rho\int_0^\pi d\theta \int_0^{ke^{\alpha\theta}} (x_1^2+x_2^2+\underbrace{x_3^2}_{0}-x_1^2)\eta\, d\eta = \rho\int_0^\pi d\theta\int_0^{ke^{\alpha\theta}} x_2^2\, \eta\, d\eta$$

$$= \rho\int_0^\pi d\theta \sin^2\theta \int_0^{ke^{\alpha\theta}} \eta^3 d\eta = \rho\int_0^\pi d\theta \sin^2\theta\, \frac{k^4 e^{4\alpha\theta}}{4}$$

$$= \frac{\rho k^4}{4}\int_0^\pi d\theta \sin^2\theta\, e^{4\alpha\theta} \overset{(E.18b)}{=} \frac{\rho k^4}{4}\left\{\frac{e^{4\alpha\theta}}{(4\alpha)^2+4}\left(4\alpha\sin^2\theta - 2\sin\theta\cos\theta + \frac{2}{4\alpha}\right)\Bigg|_0^\pi\right\}$$

$$= \frac{\rho k^4}{4}\left\{\frac{e^{4\alpha\pi}}{(4\alpha)^2+4}\,\frac{2}{4\alpha} - \frac{1}{(4\alpha)^2+4}\cdot\frac{2}{4\alpha}\right\}$$

$$\to I_{11} = \frac{\rho k^4}{2\alpha}\left[\frac{e^{4\pi\alpha}-1}{16(4\alpha^2+1)}\right]$$

$$I_{22} = \rho\int_0^\pi d\theta\int_0^{ke^{\alpha\theta}} x_1^2\, \eta\, d\eta = \rho\int_0^\pi d\theta\cos^2\theta\int_0^{ke^{\alpha\theta}}\eta^3 d\eta$$

$$= \rho\int_0^\pi d\theta(1-\sin^2\theta)\int_0^{ke^{\alpha\theta}}\eta^3 d\eta = \rho\int_0^\pi d\theta\,\frac{(ke^{\alpha\theta})^4}{4} - I_{11}$$

$$= \frac{\rho k^4}{4}\int_0^\pi d\theta\, e^{4\alpha\theta} - I_{11} = \frac{\rho k^4}{4}\left\{\frac{e^{4\alpha\pi}-1}{4\alpha}\right\} - I_{11}$$

$$= \frac{\rho k^4}{2\alpha}\left\{\frac{e^{4\alpha\pi}-1}{8}\right\} - I_{11}$$

$$\to I_{22} = \frac{\rho k^4}{2\alpha}\left\{\frac{e^{4\pi\alpha}-1}{16(4\alpha^2+1)}\right\}\left\{8\alpha^2+1\right\}$$

$$I_{12} = -\rho\int_0^\pi d\theta\int_0^{ke^{\alpha\theta}}\eta\, d\eta\; x_1x_2 = -\rho\int_0^\pi d\theta\cos\theta\sin\theta\int_0^{ke^{\alpha\theta}}\eta^3 d\eta$$

notum $\cos\theta\sin\theta = \frac{\sin(2\theta)}{2}$

$$I_{12} = -\rho\int_0^\pi d\theta\,\frac{\sin(2\theta)}{2}\int_0^{ke^{\alpha\theta}}\eta^3 d\eta = -2\alpha I_{11}$$

$$= -\rho k^4\left[\frac{e^{4\pi\alpha}-1}{16(4\alpha^2+1)}\right]$$

notum 11-17

$$\to I_{33} = I_{11}+I_{22}$$

$$= \frac{\rho k^4}{2\alpha}\left\{\frac{e^{4\pi\alpha}-1}{16(4\alpha^2+1)}\right\}\left\{8\alpha^2+2\right\}$$

$$\to \mathbb{I} = \begin{Bmatrix} F & -2\alpha F & 0\\ -2\alpha F & F(8\alpha^2+1) & 0 \\ 0 & 0 & F(8\alpha^2+2)\end{Bmatrix}$$

$$= F\begin{Bmatrix}1 & -2\alpha & 0\\ -2\alpha & (8\alpha^2+1) & 0\\ 0 & 0 & (8\alpha^2+2)\end{Bmatrix}$$

$I_3' = I_1' + I_2'$

Eigingildin og því höfuðása hverfitregðurnar eru:

$I_1' = F\left\{(4\alpha^2+1) - 2\alpha\sqrt{4\alpha^2+1}\right\}$, $I_2' = F\left\{(4\alpha^2+1) + 2\alpha\sqrt{4\alpha^2+1}\right\}$, $I_3' = F\left\{8\alpha^2+2\right\}$

(13)

⑤ Dæmi 11-13 í bók

3-punktmassar

finnum $\mathbb{I}$, höfuðása og hverfitregðu um höfuðása

$m_1 = 3m$ í $(b,0,b)$

$m_2 = 4m$ í $(b,b,-b)$

$m_3 = 2m$ í $(-b,b,0)$

$$I_{11} = \sum_\alpha m_\alpha \left\{ x_{\alpha,2}^2 + x_{\alpha,3}^2 \right\} = 3m\{b^2\} + 4m\{2b^2\} + 2m\{b^2\}$$
$$= 13\, m b^2$$

$I_{22} = 16\, mb^2$ $\quad$ $I_{12} = -2mb^2$

$I_{33} = 15\, mb^2$ $\quad$ $I_{13} = mb^2$

$I_{23} = 4mb^2$

(14)

$$\mathbb{I} = \begin{Bmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{Bmatrix} mb^2$$

eigingildi

Hverfitregður um höfuðása er

$$I_1 = mb^2\left\{17 - \sqrt{7}\right\} \simeq 14.354\, mb^2$$

$$I_2 = mb^2\left\{17 + \sqrt{7}\right\} \simeq 19.646\, mb^2$$

$$I_3 = mb^2\left\{10\right\} = 10\, mb^2$$

með höfuðása (ó staðlaða)

1: $(1, \sqrt{7}-3, \sqrt{7}-2)b$ $\quad$ 3: $(1,1,-1)b$

2: $(1, -\sqrt{7}-3, -\sqrt{7}-2)b$

eiginvigrar

(15)

⑥ Dæmi 11-30 í bók

Viðsnúningpunktar fyrir vagg (nutation) í snúð (11.162)

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta)$$

þegar $\dot{\theta} = 0$

$$\hookrightarrow E' = V(\theta) = \frac{\left\{P_\phi - P_\psi \cos\theta\right\}^2}{2I_1 \sin^2\theta} + Mgh\cos\theta$$

$$\rightarrow \left\{P_\phi - P_\psi \cos\theta\right\}^2 + 2I_1 Mgh \sin^2\theta\cos\theta - E' 2I_1 \sin^2\theta = 0$$

$$\left\{P_\phi - P_\psi \cos\theta\right\}^2 + 2I_1 Mgh(1-\cos^2\theta)\cos\theta - E' 2I_1(1-\cos^2\theta) = 0$$

(16)

$$\left\{2MghI_1\right\}\cos^3\theta - \left\{2E'I_1 + P_\psi^2\right\}\cos^2\theta + 2\left\{P_\phi P_\psi - MghI_1\right\}\cos\theta + \left\{2E'I_1 - P_\phi^2\right\} = 0$$

3. stigs margliða í $\cos\theta$, $f(\cos\theta) = 0$

Hægt er að sýna að $f(\pm 1) < 0$ $\rightarrow$ tvær rætur finnast innan bilsins $[-1,+1]$

$\rightarrow$ ein rót verður utan þess bils, sem leiðir til þvergildrar lausnar fyrir θ því $\cos\theta \geq 1$