① Dæmi 11-01 í bók

Finna I_1, I_2, og I_3 fyrir kúlu með M og R, gegnheil kúla.
Notum kúluhnit með miðju í miðju kúlu

$$I_{ij} = \int_V dv\, \rho(\bar r)\left\{\delta_{ij}\sum_k x_k^2 - x_i x_j\right\}$$

Vegna samhverfu og kúluhnita er heppilegt að reikna

$$I_{33} = \rho\int_V dv\left(r^2 - z^2\right) = \rho\int_0^{2\pi} d\phi\int_{-1}^{+1} d(\cos\theta)\int_0^R r^2 dr\cdot r^2\left(1-\cos^2\theta\right)$$

$$= 2\pi\rho\int_0^R dr\, r^4\int_{-1}^{+1} d(\cos\theta)\left(1-\cos^2\theta\right) = 2\pi\rho\frac{R^5}{5}\cdot\frac{4}{3}$$

$$M = \frac{4\pi}{3}\rho R^3 \quad\longrightarrow\quad I_{33} = \frac{2}{5}MR^2$$

Kúlan er kúlusamhverf um miðpunkt $\to I_{11} = I_{22} = I_{33}$

Athugum I_{ij} utan hornalínu

$$I_{ij} = \rho\int dv\left\{-x_i x_j\right\} \quad \text{ef } i\neq j$$

Öll þessi heildi hverfa eins og sést í kartísum hnitum með miðju í kúlu vegna andsamhverfu um 0

Því er hverfitregðuþinurinn á hornalínuham

$$\mathbb{I} = \begin{pmatrix} I_{33} & 0 & 0 \\ 0 & I_{33} & 0 \\ 0 & 0 & I_{33}\end{pmatrix}$$

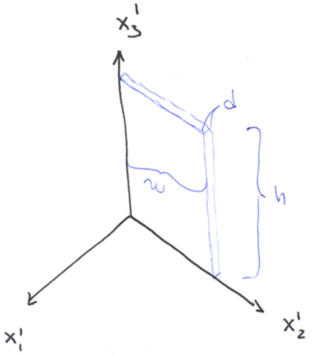
og því eru höfuðásarnir þrír hornréttir ásar um 0 og hverfitregður um þá eru $I_1 = I_2 = I_3 = \frac{2}{5}MR^2$

② Dæmi 11-08 í bók

Hurð með breidd $w = 1\,\text{m}$

Ef hún er opnuð um 90° fellur hún að stöfum á 2s.

Sýnið að lína um hjarir hlýtur að halla 3° frá lóðréttu.



$w = 1\,\text{m}$

$M = \rho whd$

$\rho = \frac{M}{whd}$

$d \ll w, h$

$$I_3 = \rho d\int_0^h dh'\int_0^w dw'\left\{h'^2 + w'^2 - h'^2\right\}$$

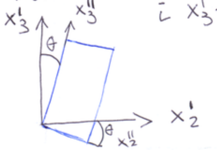
$$= \rho d\int_0^h dh'\int_0^w dw'\cdot w'^2$$

$$= \frac{M}{whd}hd\cdot\frac{w^3}{3} = \frac{1}{3}Mw^2$$

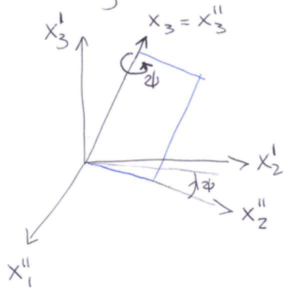
Stillum hurð með halla í upphafi, hornum Eulers

Enginn snúningur um x_3'
$\to \phi = 0$

Höllum $\to$ snúningur um $-\theta$ í x_3'-x_2'-sléttu



Snúningur um ψ um x_3''-ás $\to x_3$ (11.99)



$$\left.\begin{array}{l}\phi = 0 \\ -\theta \\ \psi\end{array}\right\} \to \lambda = \begin{pmatrix} \cos\psi & \cos\theta\sin\psi & -\sin\psi\sin\theta \\ -\sin\psi & \cos\theta\cos\psi & -\cos\psi\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Hurðin fellur að stöfum með $-\psi$ snúningi

$\omega_1 = 0,\ \omega_2 = 0$

$$\to I_3\dot\omega_3 = I_3\ddot\psi = N_3$$

Í hnitakerfi hurðar er CM:

$$\bar R = \begin{pmatrix} 0 \\ w/2 \\ h/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

ef $d \simeq 0$

Í kerfi dyrastafanna er

$$\bar R' = \lambda^{-1}\bar R = \lambda^t\bar R = \frac{1}{2}\begin{pmatrix} -w\sin\psi \\ w\cos\theta\cos\psi + h\sin\theta \\ -w\cos\psi\sin\theta + h\cos\theta \end{pmatrix}$$

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Kraftur þyngdar á hurðina í kerfi stafa

$$\bar F' = -Mg\,\hat e_3'$$

$$\to \bar N' = \bar R' \times \bar F' = -\frac{M}{2}g \begin{vmatrix} \hat e_1' & \hat e_2' & \hat e_3' \\ -w\sin2\psi, & w\cos2\psi\cos\theta + h\sin\theta, & -w\cos2\psi\sin\theta + h\cos\theta \\ 0 & 0 & 1\end{vmatrix}$$

$$= -\frac{Mg}{2}\begin{pmatrix} w\cos2\psi\cos\theta + h\sin\theta \\ w\sin2\psi \\ 0\end{pmatrix}$$

Í kerfi hurðar fast því

$$\bar N = \lambda \bar N' = -\frac{Mg}{2}\begin{pmatrix} w\cos^2 2\psi\cos\theta + h\sin\theta\cos2\psi + w\sin^2 2\psi\cos\theta \\ -h\sin\theta\sin2\psi \\ w\sin\theta\sin2\psi\end{pmatrix}$$

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$$\to N_3 = -\frac{Mg}{2}w\sin\theta\sin2\psi$$

og með hreyfijöfnunni

$$I_3 2\ddot\psi = N_3 = -\frac{Mg}{2}w\sin\theta\sin2\psi$$

$$I_3 = \frac13 Mw^2$$

$$\to \boxed{2\ddot\psi = -\frac32\frac{g}{w}\sin\theta\sin2\psi}$$

sem við þurfum að leysa

margföldum með $2\dot\psi$

$$2\dot\psi\,2\ddot\psi = \left(-\frac32\frac{g}{w}\sin\theta\right)2\dot\psi\sin2\psi$$

heildum óákveðið m.t.t. t

$$\frac12 (2\dot\psi)^2 = \left(\frac{3g}{2w}\sin\theta\right)\cos2\psi + C_1$$

Opnuðum 90°

$\cos(2\psi(0)) = 0$

$2\dot\psi(0) = 0$

$\to C_1 = 0$

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$$\to 2\dot\psi = \pm\sqrt{\frac{3g}{w}\sin\theta\cos2\psi}$$

hurðin fellur að stöfum að $2\dot\psi < 0$ þegar $\cos2\psi > 0$

$$\int_{\pi/2}^{0} \frac{d2\psi}{\sqrt{\cos2\psi}} = -\sqrt{\frac{3g}{w}\sin\theta}\int_0^T dt$$

tíminn sem var gefinn sem 2s

$$= -T\sqrt{\frac{3g}{w}\sin\theta}$$

$$\to \int_0^{\pi/2}\frac{d2\psi}{\sqrt{\cos2\psi}} = T\sqrt{\frac{3g}{w}\sin\theta}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{(GR: 8.384.1)}$$

$$B\left(\tfrac14,\tfrac14\right) = \frac{\Gamma(\frac14)\Gamma(\frac14)}{\Gamma(1/2)}$$

$$\int_0^{\pi/2}\cos^{\mu-1}x\,dx = 2^{\mu-2}B\left(\frac{\mu}{2},\frac{\mu}{2}\right)$$

(GR: 3.621.1)

hér $\mu = \frac12$

$$\to \int_0^{\pi/2}\cos^{-1/2}x\,dx = 2^{-3/2}B\left(\tfrac14,\tfrac14\right)$$

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$$\to \int_0^{\pi/2}\cos^{-1/2}x\,dx = \frac{(\Gamma(\frac14))^2}{\Gamma(\frac12)}\left(\frac{1}{\sqrt2}\right)^3 \approx 2.6221 \simeq T\sqrt{\frac{3g}{w}\sin\theta}$$

$$\to \sin\theta \approx \frac{w}{3gT^2}\cdot(2.621)^2$$

$g = 9.81\ m/s^2$

$$\theta \approx \arcsin\left(\frac{w}{3gT^2}(2.621)^2\right) \simeq 0{,}0584 \text{ rad} \simeq 3.35^\circ$$

3) Dæmi 11-06 í bók

Hvernig getum við greint á milli?

Ein kúla gegnheil

Önnur kúluskel

Sami M og R

Finnum hverfitregðuna um miðju þeirra

Hér að undan sáum við að fyrir gegnheila kúlu fæst $I_s = \frac25 MR^2$ fyrir ás í gegnum miðju.

Kúluskel

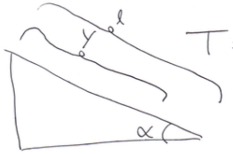
Gefum okkur að hún sé þunn $\to$ $M = 4\pi R^2 \cdot \rho_s$ Flatar þéttleiki massans

Reiknum fyrir $\hat{e}_z$-ás um miðju (það er höfuðás)

$$I_h = \rho_s \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \left\{ R^2 - z^2 \right\} R^2 \qquad z = R\cos\theta$$

$$= \rho_s \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^3\theta \cdot R^4 = \frac{8}{3}\pi \rho_s R^4 = \frac{2}{3} M R^2$$

Ef við látum kúlurnar rúlla niður skáplan má búast við mismuni.



$$T = \frac{M}{2}\dot{y}^2 + \frac{I}{2}\dot{\theta}^2, \qquad U = Mg(l-y)\sin\alpha, \qquad y = R\theta$$

Sleppum

$$L = \frac{M}{2}\dot{y}^2 + \frac{I}{2R^2}\dot{y}^2 + Mgy\sin\alpha$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \qquad L = \left\{ \frac{M}{2} + \frac{I}{2R^2} \right\} \dot{y}^2 + Mgy\sin\alpha$$

$$\frac{\partial L}{\partial y} = Mg\sin\alpha$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}\left(\dot{y}\left\{ M + \frac{I}{R^2} \right\} \right) = \ddot{y}\left\{ M + \frac{I}{R^2} \right\}$$

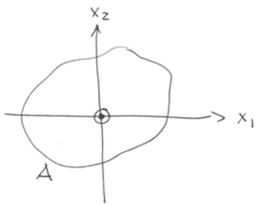
$$\to Mg\sin\alpha - \ddot{y}\left\{ M + \frac{I}{R^2} \right\} = 0 \to \ddot{y} = \frac{gMR^2\sin\alpha}{\left\{ MR^2 + I \right\}}$$

$\to$ Kúlan með minni I hraðast meir niður skáplanið

$$I_h = \frac{2}{3}MR^2 \qquad I_s = \frac{2}{5}MR^2$$

$\frac{2}{5} < \frac{2}{3}$ $\to$ gegnheila kúlan fer hraðar

(4) Dæmi 11-17 í bók



Miðum við o

Einsleit plata í x_1-x_2-sléttu

$$I_{11} = \rho_s \int_A dx_1 dx_2 \left\{ r^2 - x_1^2 \right\}$$

$$= \rho_s \int_A dx_1 dx_2 \left\{ x_2^2 + x_3^2 \right\} = \rho_s \int dx_1 dx_2 \cdot x_2^2 \equiv A$$

því $x_3 = 0$ í sléttunni

$$I_{22} = \rho_s \int dx_1 dx_2 \left\{ r^2 - x_2^2 \right\}$$

$$= \rho_s \int dx_1 dx_2 \left\{ x_1^2 + x_3^2 \right\} = \rho \int dx_1 dx_2 \cdot x_1^2 \equiv B$$

$$I_{33} = \rho_s \int dx_1 dx_2 \left\{ r^2 - x_3^2 \right\} = \rho \int dx_1 dx_2 \left\{ x_1^2 + x_2^2 \right\} = A + B$$

$$I_{13} = I_{31} = \rho_s \int dx_1 dx_2 \left\{ -x_1 x_3 \right\} = 0$$

$$I_{23} = I_{32} = \rho_s \int dx_1 dx_2 \left\{ -x_2 x_3 \right\} = 0$$

$$I_{12} = I_{21} = \rho_s \int dx_1 dx_2 \left\{ -x_1 x_2 \right\} = -C$$

$$\Rightarrow \mathbb{I} = \begin{Bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{Bmatrix}$$

Hér má því finna almenna jöfnu sem aðeins inniheldur A, B, og C fyrir höfuðásunum þó lögunin sé ekki alveg gefin

(13)

(5) Dæmi 11-18 í bók

Ef í dæminu að undan ásarnum er snúið um θ um x_3-ás finna $\mathbb{I}'$

Aðeins snúningur um x_3'-ás – (11.91) $\rightarrow \lambda = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\mathbb{I}' = \lambda \mathbb{I} \lambda^t$$

$$= \begin{Bmatrix} A\cos^2\theta - C\sin(2\theta) + B\sin^2\theta & -C\cos(2\theta) + \frac{B-A}{2}\sin(2\theta) & 0 \\ -C\cos(2\theta) + \frac{B-A}{2}\sin(2\theta) & A\sin^2\theta + C\sin(2\theta) + B\cos^2\theta & 0 \\ 0 & 0 & A+B \end{Bmatrix}$$

$$= \begin{Bmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A+B \end{Bmatrix}$$

með
$A' = A\cos^2\theta - C\sin(2\theta) + B\sin^2\theta$
$B' = A\sin^2\theta + C\sin(2\theta) + B\cos^2\theta$
$C' = C\cos(2\theta) - \frac{B-A}{2}\sin(2\theta)$

(14)

fyrir x_1 og x_2 að verða höfuðása þarf að gilda

$$C' = C\cos(2\theta) - \frac{B-A}{2}\sin(2\theta) = 0$$

$$\rightarrow \frac{\sin(2\theta)}{\cos(2\theta)} = \tan(2\theta) = \left(\frac{2C}{B-A}\right)$$

$$\rightarrow \theta = \frac{1}{2}\arctan\left(\frac{2C}{B-A}\right)$$

---

(6) Dæmi 11-16 í bók

$$\mathbb{I} = \begin{Bmatrix} \frac{A+B}{2} & \frac{A-B}{2} & 0 \\ \frac{A-B}{2} & \frac{A+B}{2} & 0 \\ 0 & 0 & C \end{Bmatrix}$$

(15)

Snúa um θ um x_3-ás

$$\lambda = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{I}' = \lambda \mathbb{I} \lambda^t$$

$$= \begin{Bmatrix} \frac{A+B}{2} + (A-B)\cos\theta\sin\theta & \frac{A-B}{2}\cos^2\theta - \frac{A-B}{2}\sin^2\theta & 0 \\ -\frac{A-B}{2}\sin^2\theta + \frac{A-B}{2}\cos^2\theta & \frac{A+B}{2} - \frac{A-B}{2}\cos\theta\sin\theta & 0 \\ 0 & 0 & C \end{Bmatrix}$$

ef $\theta = \frac{\pi}{4}$

$\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

þá fæst $\begin{Bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{Bmatrix}$