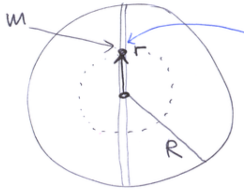
(1)

① Dæmi 5-15 í bók.



ρ er fasti, þéttleiki jarðar í þessu líkani
Lögmál Gauß → aðeins massinn innan geislans r, $M(r)$, veldur krafti á ögnina að miðju

$$F(r) = -\frac{GmM(r)}{r^2}, \quad M(r) = \frac{4}{3}\pi r^3 \rho$$

$$= -Gmr\frac{4\pi\rho}{3} = -G\frac{m4\pi\rho}{3}r = m\ddot{r}$$

$$\to \ddot{r} + G\frac{4\pi\rho}{3}r = 0 \quad \text{eða} \quad \boxed{\ddot{r} + \omega_0^2 r = 0}$$

Hreintóna sveifla með $\omega_0 = \sqrt{\frac{4\pi G\rho}{3}} = 2\pi\sqrt{\frac{G\rho}{3\pi}}$

Lota $\boxed{\tau = \frac{2\pi}{\omega_0} = \sqrt{\frac{3\pi}{G\rho}}}$

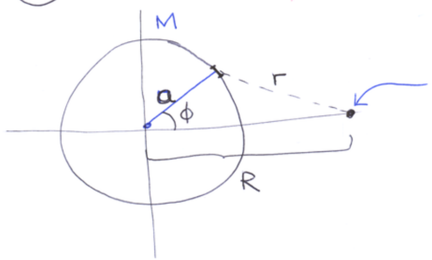
(2)

$\rho = 5.514\ \text{g/cm}^3$, $G = 6{,}674\cdot 10^{-8}\ \frac{\text{cm}^3}{\text{g}\,\text{s}^2}$

$$\to \tau = \sqrt{\frac{3\pi}{5.514\cdot 6{,}674\cdot 10^{-8}}}\ \text{s} \simeq 5061\ \text{s} \approx 84\ \text{mín}$$

Ef $\rho = \rho(r)$ má finna $M(r) = 4\pi\int_0^r x^2 dx\, \rho(x)$

② dæmi 5-9 í bók. Reikna mættið Φ fyrir punkt í sléttu hringsins



$$d\Phi = -G\frac{dM}{r} = -G\frac{\rho_\ell d\ell}{r}$$

$$d\ell = a\,d\phi$$

$$r = \sqrt{R^2 + a^2 - 2aR\cos\phi}$$

$$\rho_\ell = \frac{M}{2\pi a}$$

(3)

$$\Phi(R) = -\frac{GM}{2\pi}\int_0^{2\pi}\frac{d\phi}{\sqrt{R^2+a^2-2aR\cos\phi}} = -\frac{GM}{2\pi R}\int_0^{2\pi}\frac{d\phi}{\sqrt{1+\left(\frac{a}{R}\right)^2 - 2\left(\frac{a}{R}\right)\cos\phi}}$$

Athugum liðinn fyrir $\left(\frac{a}{R}\right) \ll 1$, notum $\frac{1}{\sqrt{1+x}} \simeq 1 - \frac{x}{2} + \frac{3x^2}{8} + \dots$

$$\to \Phi(R) \simeq -\frac{GM}{2\pi R}\int_0^{2\pi} d\phi\left[1 - \frac{1}{2}\left\{\left(\frac{a}{R}\right)^2 - 2\left(\frac{a}{R}\right)\cos\phi\right\} + \frac{3}{8}\left\{\left(\frac{a}{R}\right)^2 - 2\left(\frac{a}{R}\right)\cos\phi\right\}^2 + \dots\right]$$

höldum $\left(\frac{a}{R}\right)^2$-liðum

$$\to \boxed{\Phi(R)} \simeq -\frac{GM}{2\pi R}\int_0^{2\pi} d\phi\left\{1 - \frac{1}{2}\left(\frac{a}{R}\right)^2 + \overbrace{\left(\frac{a}{R}\right)\cos\phi}^{\to 0} + \frac{3}{2}\left(\frac{a}{R}\right)^2\cos^2\phi\right\}$$

$$= -\frac{GM}{2\pi R}\left\{2\pi - \pi\left(\frac{a}{R}\right)^2 + \frac{3}{2}\left(\frac{a}{R}\right)^2\pi\right\}$$

$$= \boxed{-\frac{GM}{R}\left\{1 + \frac{1}{4}\left(\frac{a}{R}\right)^2\right\}} \quad \text{ef} \quad \left(\frac{a}{R}\right) \ll 1$$

(4)

③ Kúla með massadreifingu $\rho(r)$

Við vitum að $\bar{g} = -\nabla\Phi$ og $\nabla^2\Phi = 4\pi G\rho(r)$

Ef ρ er aðeins háð r → Φ er aðeins fall af r

Í kúluhnitum jafna Poissons

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = 4\pi G\rho(r)$$

eins er $\bar{g} = -\nabla\Phi = -\hat{e}_r\frac{\partial}{\partial r}\Phi(r)$

Ef $\bar{g}$ er óháð r, þá er $\frac{\partial}{\partial r}\Phi(r) = -g_0 = \text{fasti}$

Ef Φ er aðeins háð r, þá er g ekki háð ϕ eða θ

5

$\frac{\partial}{\partial r}\Phi(r) = g_0$, notum í jöfnu Poissons

$$\underbrace{-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g_0\right)} = 4\pi G\rho(r)$$

$$-g_0\frac{2}{r^2}r = -g_0\frac{2}{r} \;\longrightarrow\; -g_0\frac{2}{r} = 4\pi G\rho(r)$$

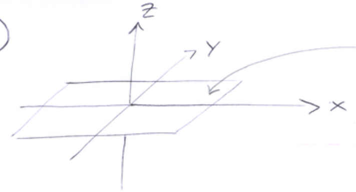
$$\longrightarrow \rho(r) = -\frac{g_0}{2\pi G r}$$

Við búumst því við að $g_0 < 0$

Í miðju kúlunnar er sérstöðupunktur fyrir massann, af sömu tegund og punktmassi hefði, ef ekki þá hefði g þurft að vera háð r þ.a. $\bar{g}(r) \to 0$ þegar $r \to 0$

6

4



massaþéttleiki ρ_s

$[\rho_s] = \frac{M}{L^2}$ ← massi á flöt

Finna þyngdarmættið Φ og sviðið $\bar{g}$

$$\nabla^2\Phi = 4\pi G\rho, \qquad \bar{g} = -\bar{\nabla}\Phi$$

Notum lögmál Gauß

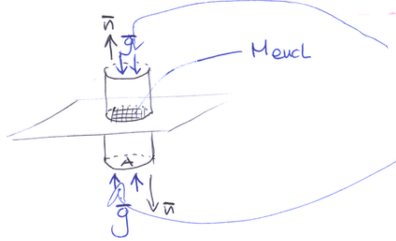
$$\oint_S \bar{n}\cdot\bar{g}\,da = -4\pi G\int_V \rho\,dv = -4\pi G M_{encl}$$

Það er aðdráttur að sléttunni, samhverfur hægra og vinstra megin, vegna samhverfu getur $\bar{g}$ aðeins verið hornrétt á sléttuna.

$g \rightrightarrows | \leftleftarrows g$

7

Hugsum okkur lokað Gauß yfirborð sem sívalning þvert á sléttuna



Eina flæðið er um topp og botn sívalningsins, ekkert um bogna flötinn! Flæðið er óháð hæð sívalnings

$$\oint_S \bar{n}\cdot\bar{g}\,da = -2Ag$$

$$-4\pi G M_{encl} = -4\pi G(\rho_s A)$$

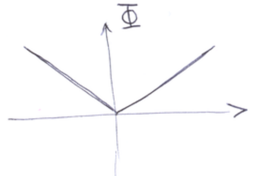
$$\longrightarrow g = 2\pi G\rho_s$$

því fæst $\boxed{\bar{g} = -\hat{e}_z\frac{z}{|z|}2\pi G\rho_s}$

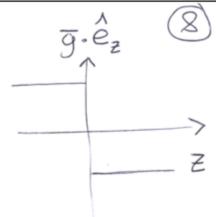
fæst þyngdarsvið óháð fjarlægð frá sléttu sitthvoru megin með stefnu að sléttu

8

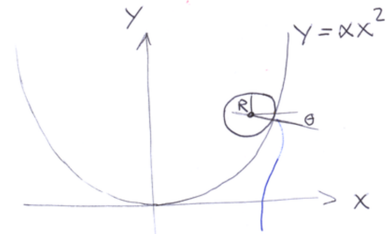
$$\bar{g} = -\bar{\nabla}\Phi \;\longrightarrow\; \Phi = |z|\,2\pi G\rho_s$$



brot í Φ og þrep í $\bar{g}$ vegna massasléttu



5 Dæmi 6-11 í bók



Hvernig eru skilyrðin fyrir því að skífan velti þ.a. hún snerti fleygbogann aðeins í einum punkti

$$ds = R\,d\theta$$

$$\hookrightarrow = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(9)

$y = \alpha x^2 \rightarrow \frac{dy}{dx} = 2\alpha x$

$\rightarrow ds = \sqrt{1+(2\alpha x)^2}\,dx = R\,d\theta$, heildum

$R\int d\theta = \int \sqrt{1+(2\alpha x)^2}\,dx$

$\boxed{C + R\theta = \frac{x\sqrt{1+(2\alpha x)^2}}{2} + \frac{\mathrm{ArSinh}(2\alpha x)}{4\alpha}}$ ← staða $\theta(x)$

$\alpha > 0$

heildunarfasti ákvarðast af upphafsskilyrðum

Fyrir fall $y=f(x)$ er sveigjugeisli (radius of curvature)

$\frac{1}{r_0} = \frac{|y''|}{(1+(y')^2)^{3/2}}$

$\rightarrow r_0 \geq \frac{1}{2\alpha}$ fyrir alla punkta (x,y) á fleygboganum

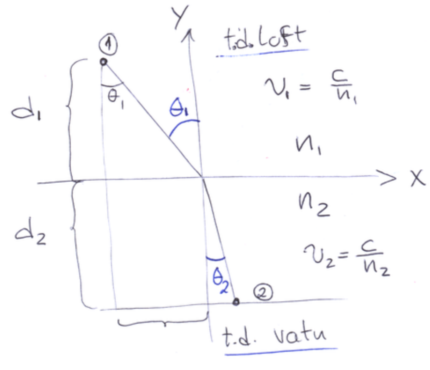
$= \frac{2\alpha}{(1+(2\alpha x)^2)^{3/2}}$ fyrir fleygbogann

(10)

$\rightarrow$ skífan veltur alls staðar ef $R < r_0 \geq \frac{1}{2\alpha}$

$\rightarrow \boxed{R < \frac{1}{2\alpha}}$

(6) Dæmi 6-7 í bók



$n_2 > n_1$

Lágmarka tíma til þess að leiða út $n_1 \sin\theta_1 = n_2 \sin\theta_2$

$\frac{ds}{dt} = v \rightarrow dt = \frac{ds}{v}$

$\int_1^2 dt = \int_1^2 \frac{ds}{v}$

$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

(11)

$\rightarrow t_2 - t_1 = \Delta t = \int_1^2 \frac{\sqrt{1+(y')^2}}{v}\,dx$

Notum Euler $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$, $f(y,y') = \frac{\sqrt{1+(y')^2}}{v}$

$v = v(y)$, en $\frac{dv}{dy} = 0$ nema í punktinum $y = 0$

$\left.\begin{aligned} \frac{\partial f}{\partial y} &= 0 \\ \frac{\partial f}{\partial y'} &= \frac{y'}{v\sqrt{1+(y')^2}} \end{aligned}\right\} \rightarrow \frac{d}{dx}\left\{\frac{y'}{v\sqrt{1+(y')^2}}\right\} = 0$

Vitum að $v_i = \frac{c}{n_i}$

$\left.\begin{aligned} \frac{dy_i}{dx} = -\tan\theta_i = y' \end{aligned}\right\} \quad \frac{-\tan\theta_i \cdot n_i}{c\sqrt{1+\tan^2\theta_i}} = C_1$ ← fasti

(12)

$1 + \tan^2\theta = \frac{1}{\cos\theta}$

$\frac{-\tan\theta_i \cdot n_i}{c\sqrt{1+\tan^2\theta_i}} = C_1 \rightarrow \underbrace{-\frac{n_i}{c}\sin\theta_i = C_1}_{\text{Lögmál Snells}}$ fasti