(1)

1. Dæmatímar

Bolta hent beint upp með ferð v_0 í föstu þyngdarsviði. Hversu langt upp fer hann. Viðnámskraftur $f \sim v^2$

Hreyfijafnan

$$m\ddot z = -mDv^2 - mg$$

Vegna viðnámsliðar er heppilegt að umrita fyrir v í stað z

$$\ddot z = \frac{d^2z}{dt^2} = \frac{d}{dt}\left\{\frac{dz}{dt}\right\} = \frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz}$$

(2)

Hreyfijafnan er því

$$mv\frac{dv}{dz} = -mDv^2 - mg$$

$$\to \quad \frac{v\,dv}{Dv^2+g} = -dz$$

$$\int_{v_0}^{v}\frac{v'dv'}{D(v')^2+g} = -\int_0^{z}dz' \quad \to \quad \frac{1}{2D}\ln(D(v')^2-g)\Big|_{v_0}^{v} = -z$$

$$\to \quad z = -\frac{1}{2D}\ln\left\{\frac{Dv^2+g}{Dv_0^2+g}\right\} = \frac{1}{2D}\ln\left\{\frac{Dv_0^2+g}{Dv^2+g}\right\}$$

Hæsti punkturinn er þegar $v = 0$

$$\to \quad z_{max} = \frac{1}{2D}\ln\left\{\frac{Dv_0^2+g}{g}\right\}$$

(3)

$$z_{max} = \frac{1}{2D}\ln\left\{\frac{Dv_0^2}{g}+1\right\}$$

$$= \frac{1}{2D}\left\{\frac{Dv_0^2}{g} - \frac{(Dv_0^2)^2}{2g^2} + o(D^3)\right\}$$

$$\to \quad z_{max}(D\to 0) = \frac{v_0^2}{2g}$$

best er graf af

$$\frac{2z_m g}{v_0^2} = \frac{g}{Dv_0^2}\ln\left\{\frac{Dv_0^2}{g}+1\right\}$$

$$[D] = \frac{1}{L}$$

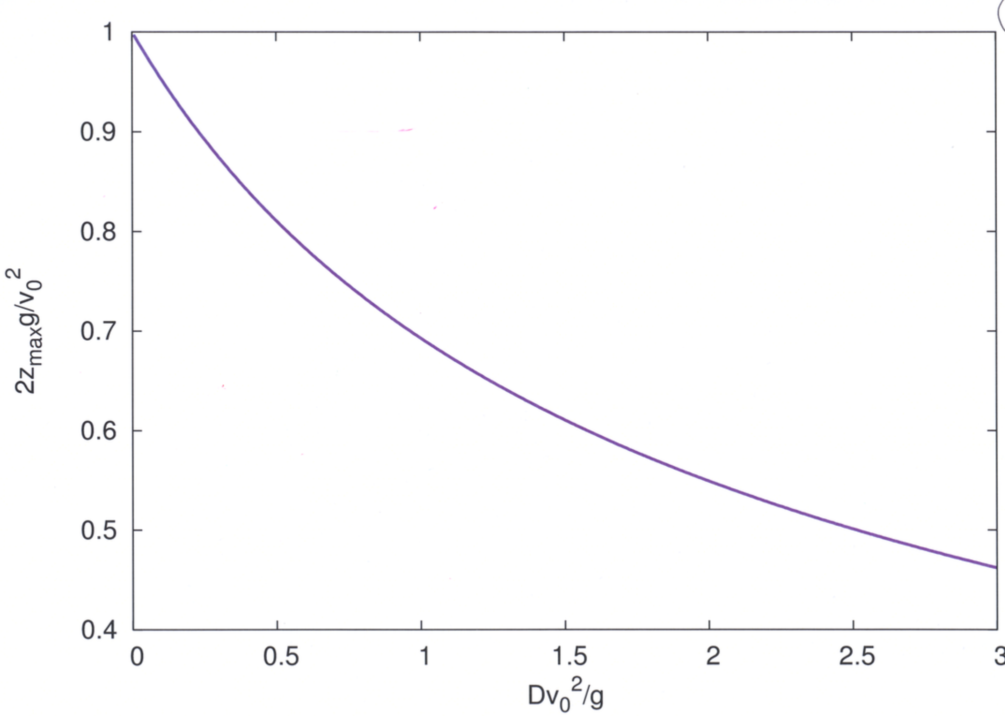
$$\to \left[\frac{Dv_0^2}{g}\right] = 1$$

og $\frac{2z_m g}{v_0^2}(D\to 0) = 1$

(2) Kraftur á ögn í kúluhnitum $F(r,\theta,\phi)$. Hvernig er hreyfijafnan?

Vandinn hér er að $\hat r, \hat\theta$ og $\hat\phi$ eru tímaháðir, það á ekki við fyrir Kartísk hnit. Sjá (1.14) í bók

(3b)



Staðsetning agnarinnar er $\bar r = r\hat e_r$ eða $r\hat r$
því er hreyfijafna hennar

$$m\ddot{\bar r} = \bar F$$

$$\bar v = \dot{\bar r} = \dot r\hat e_r + r\dot{\hat e}_r$$

Vitum að einingarvigrarnir í Kartískum hnitum eru óháðir tíma —> leiðum

$$\hat e_r = \hat e_1\sin\theta\cos\phi + \hat e_2\sin\theta\sin\phi + \hat e_3\cos\theta$$

$$\hat e_\theta = \hat e_1\cos\theta\cos\phi + \hat e_2\cos\theta\sin\phi - \hat e_3\sin\theta$$

$$\hat e_\phi = -\hat e_1\sin\phi + \hat e_2\cos\phi$$

$$\to \dot{\hat e}_r = \hat e_1\{\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi\} + \hat e_2\{\dot\theta\cos\theta\sin\phi + \dot\phi\sin\theta\cos\phi\} - \hat e_3\{\dot\theta\sin\theta\}$$

$$\to \boxed{\dot{\hat e}_r = \hat e_\theta\dot\theta + \hat e_\phi\dot\phi\sin\theta}$$

eins fæst

$$\boxed{\begin{aligned}\dot{\hat e}_\theta &= -\hat e_r\dot\theta + \hat e_\phi\dot\phi\cos\theta\\ \dot{\hat e}_\phi &= -\hat e_r\dot\phi\sin\theta - \hat e_\theta\dot\phi\cos\theta\end{aligned}}$$

$$\to \bar v = \dot{\bar r} = \dot r\hat e_r + r\dot{\hat e}_r = \dot r\hat e_r + r\hat e_\theta\dot\theta + r\hat e_\phi\dot\phi\sin\theta$$

$$\bar a = \dot{\bar v} = \ddot r\hat e_r + \dot r\dot{\hat e}_r + \dot r\hat e_\theta\dot\theta + r\dot{\hat e}_\theta\dot\theta + r\hat e_\theta\ddot\theta + \dot r\hat e_\phi\dot\phi\sin\theta + r\dot{\hat e}_\phi\dot\phi\sin\theta + r\hat e_\phi\dot\phi\dot\theta\cos\theta + r\hat e_\phi\ddot\phi\sin\theta$$

$$\bar a = \ddot r\hat e_r + \dot r\{\hat e_\theta\dot\theta + \hat e_\phi\dot\phi\sin\theta\} + \dot r\hat e_\theta\dot\theta + r\{-\hat e_r\dot\theta + \hat e_\phi\dot\phi\cos\theta\}\dot\theta + r\hat e_\theta\ddot\theta + \dot r\hat e_\phi\dot\phi\sin\theta + r\{-\hat e_r\dot\phi\sin\theta - \hat e_\theta\dot\phi\cos\theta\}\dot\phi\sin\theta + r\hat e_\phi\dot\phi\dot\theta\cos\theta + r\hat e_\phi\ddot\phi\sin\theta$$

$$\bar a = \hat e_r\left[\ddot r - r\dot\theta^2 - r\dot\phi^2\sin^2\theta\right] + \hat e_\theta\left[r\ddot\theta + 2\dot r\dot\theta - r\dot\phi^2\cos\theta\sin\theta\right] + \hat e_\phi\left\{r\ddot\phi\sin\theta + 2\dot r\dot\phi\sin\theta + 2r\dot\phi\dot\theta\cos\theta\right\}$$

$$= \frac{1}{m}\left\{\hat e_r F_r + \hat e_\theta F_\theta + \hat e_\phi F_\phi\right\}$$

3

Bát ýtt af stað með v_0, $F = -F_0 e^{+\beta v}$ viðnámskr.

Hreyfijafna $ma = m\ddot x = \boxed{m\dot v = -F_0 e^{+\beta v}}$

$$\to m\frac{dv}{dt} + F_0 e^{+\beta v} = 0, \quad v(0) = v_0$$

Aðgreinanleg afleiðujafna

$$dt = -\frac{m}{F_0}e^{-\beta v}dv \to \int_0^t dt' = -\frac{m}{F_0}\int_{v_0}^{v(t)} dv\, e^{-\beta v}$$

$$\to t = +\frac{m}{F_0\beta}\left\{e^{-\beta v(t)} - e^{-\beta v_0}\right\}$$

$$+\left\{\frac{F_0\beta t}{m} + e^{-\beta v_0}\right\} = +e^{-\beta v(t)} \to -\beta v(t) = \ln\left\{\frac{F_0\beta t}{m} + e^{-\beta v_0}\right\}$$

(8)

$$\boxed{v(t) = -\frac{1}{\beta}\ln\left\{\frac{F_0\beta t}{m} + e^{-\beta v_0}\right\}}$$

Finnum t_m þegar $v(t_m) = 0$

$$0 = -\frac{1}{\beta}\ln\left\{\frac{F_0\beta t_m}{m} + e^{-\beta v_0}\right\}$$

$$\ln(1) = 0 \longrightarrow \frac{F_0\beta t_m}{m} + e^{-\beta v_0} = 1$$

eða

$$t_m = \frac{m}{F_0\beta}\left\{1 - e^{-\beta v_0}\right\}$$

max tíminn, en eigum eftir að finna vegalengdina

$$\frac{dx}{dt} = -\frac{1}{\beta}\ln\left\{\frac{F_0\beta t}{m} + e^{-\beta v_0}\right\}$$

(9)

$$\int_0^x dx' = -\frac{1}{\beta}\int_0^t dt' \ln\left\{\frac{F_0\beta t'}{m} + e^{-\beta v_0}\right\}$$

notum $\int dx \ln(a+bx) = \left\{x + \frac{a}{b}\right\}\ln(a+bx) - x$ (GR: 2.729)

(eða maxima, en þá þarf að henda einum aukafasta)

$$x = -\frac{1}{\beta}\left\{\left\{t + \frac{e^{-\beta v_0} m}{F_0\beta}\right\}\ln\left\{e^{-\beta v_0} + \frac{F_0\beta t}{m}\right\} - t - \frac{e^{-\beta v_0} m}{F_0\beta}(-\beta v_0)\right\}$$

$$= -\frac{1}{\beta}\left\{\left\{t + \frac{e^{-\beta v_0} m}{F_0\beta}\right\}\ln\left\{e^{-\beta v_0} + \frac{F_0\beta t}{m}\right\} - t + \frac{e^{-\beta v_0} m}{F_0\beta}\beta v_0\right\}$$

(10)

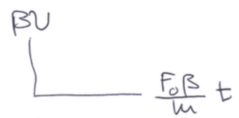
Vegalengdin verður $x(t_m)$

$$x(t_m) = -\frac{1}{\beta}\left\{\frac{m}{F_0\beta}\overbrace{\ln\left[e^{-\beta v_0} + 1 - e^{-\beta v_0}\right]}^{=0} - \frac{m}{F_0\beta}\left[1 - e^{-\beta v_0}\right] + \frac{m}{F_0\beta}e^{-\beta v_0}\beta v_0\right\}$$

$$\boxed{x(t_m) = +\frac{m}{\beta^2 F_0}\left\{1 - e^{-\beta v_0}(1 + \beta v_0)\right\}}$$

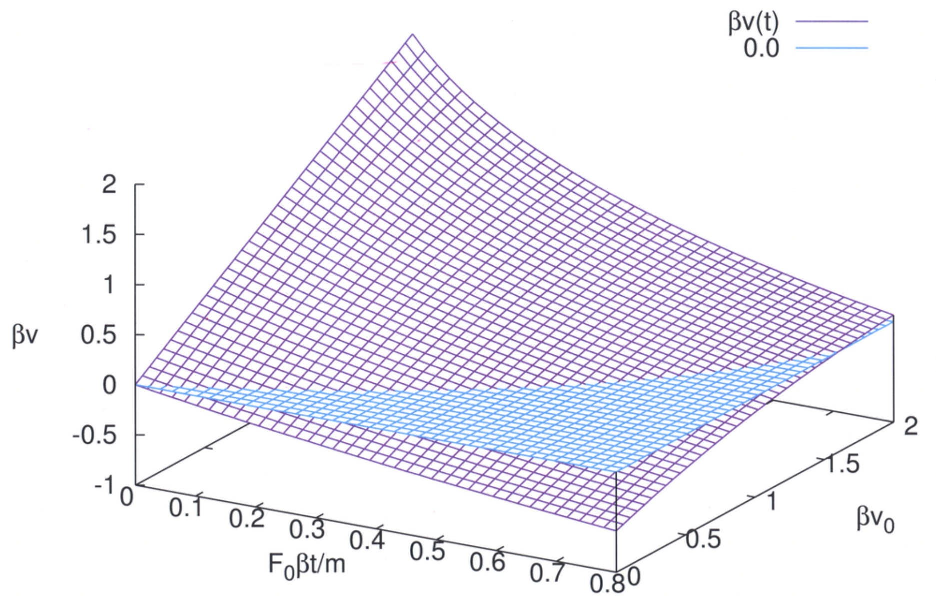
Gröf

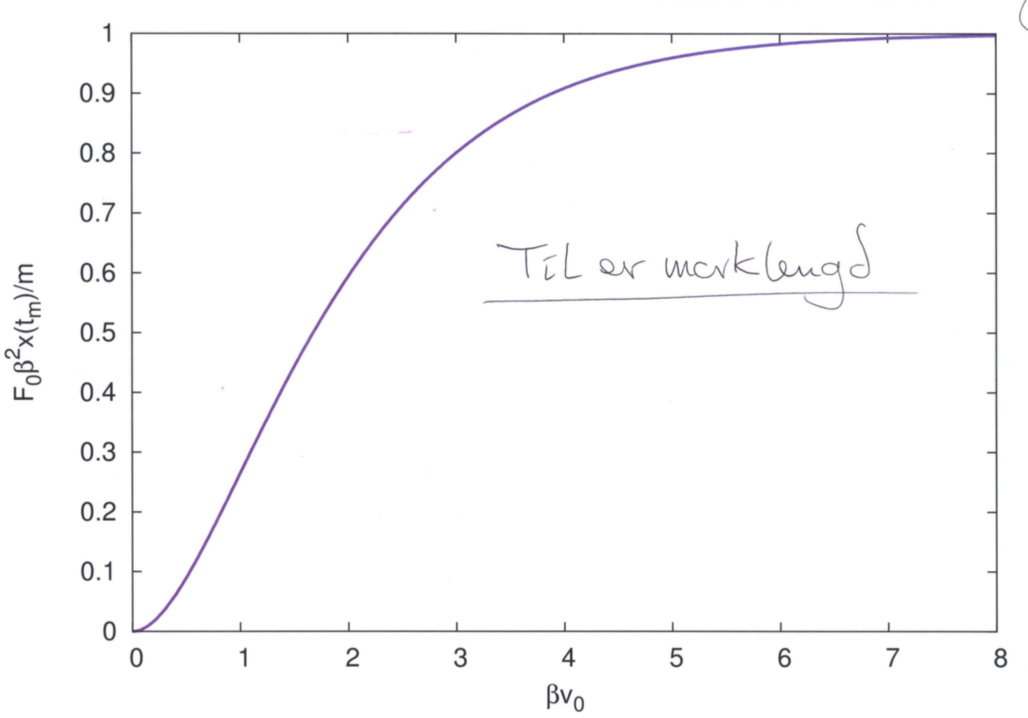
$$\beta v(t) = -\ln\left\{\frac{F_0\beta t}{m} + e^{-\beta v_0}\right\} \longrightarrow$$



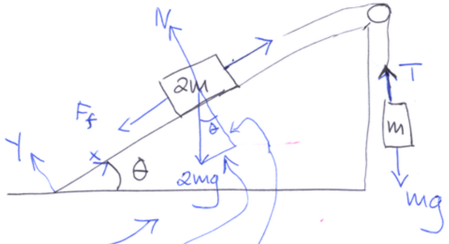
$$\frac{F_0\beta^2}{m}x(t_m) = \left\{1 - e^{-\beta v_0}(1 + \beta v_0)\right\} \longrightarrow$$

(11)





4



fyrir hvaða θ fæst jafnur hraði fyrir massana

$$mg - T = ma$$

$$\rightarrow T = m(g-a)$$

y-hnit: $N - 2mg\cos\theta = m\ddot{y} = 0$

$$\rightarrow N = 2mg\cos\theta$$

x-hnit: $T - 2mg\sin\theta - 2\mu_k mg\cos\theta = ma$

Við leitum að θ_0 sem gefur $a=0$

$$\rightarrow (g-\underset{\shortparallel\atop 0}{a}) - 2g\sin\theta_0 - 2\mu_k g\cos\theta_0 = 0$$

$$g - 2g\sin\theta_0 - 2\mu_k g\cos\theta_0 = 0$$

$$\frac{1}{2} - \sin\theta_0 - \mu_k\cos\theta_0 = 0$$

$$\frac{1}{2} - \sin\theta_0 - \mu_k\sqrt{1-\sin^2\theta} = 0$$

$$\frac{1}{2} - \sin\theta_0 = \mu_k\sqrt{1-\sin^2\theta_0} \rightarrow \left(\frac{1}{2} - \sin\theta_0\right)^2 = \mu_k^2\left(1-\sin^2\theta_0\right)$$

$$\rightarrow \sin^2\theta_0 + \frac{1}{4} - \sin\theta_0 = \mu_k^2\left(1 - \sin^2\theta_0\right)$$

$$(1+\mu_k^2)\sin^2\theta_0 - \sin\theta_0 + \left(\frac{1}{4} - \mu_k^2\right) = 0$$

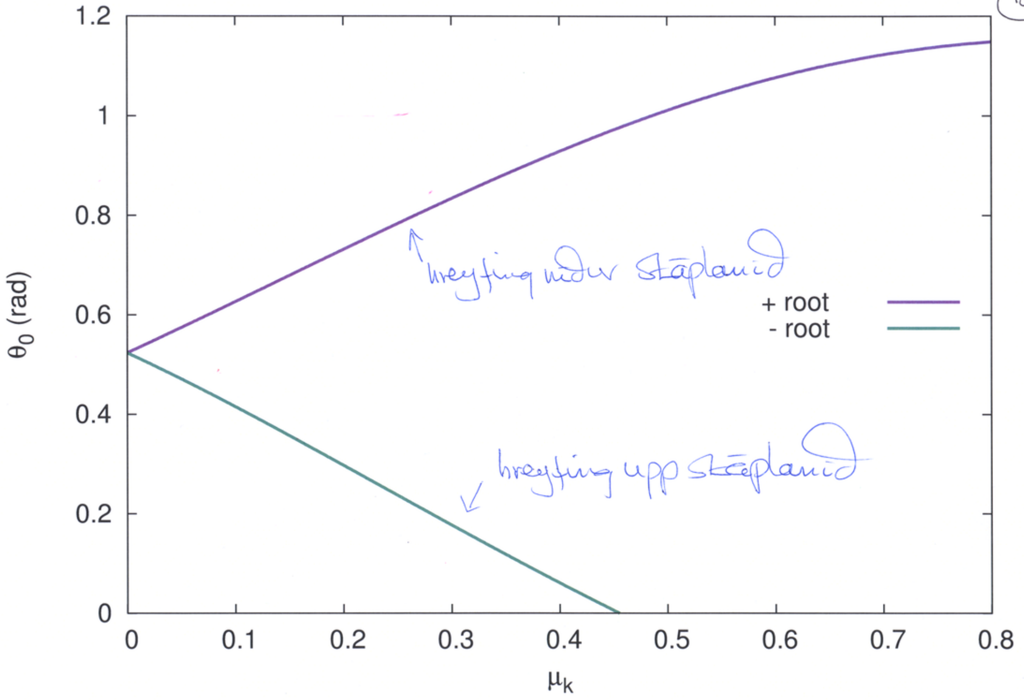
$$\rightarrow \sin\theta_0 = \frac{1 \pm \sqrt{1 - 4(1+\mu_k^2)\left(\frac{1}{4}-\mu_k^2\right)}}{2(1+\mu_k^2)} = \frac{1 \pm \mu_k\sqrt{3+4\mu_k^2}}{2(1+\mu_k^2)}$$

$$\theta_0 = \arcsin\left\{\frac{1 \pm \mu_k\sqrt{3+4\mu_k^2}}{2(1+\mu_k^2)}\right\}$$

tvær rætur

eini staðurinn þ.s. μ_k kemur fyrir

hreyfing í tvær stefnur



(5) 2-21 í bók

Kasthreyfing í x-z sléttu

$$\left.\begin{aligned} x &= v_0 t \cos\theta \\ z &= v_0 t \sin\theta - \tfrac{1}{2} g t^2 \end{aligned}\right\} \quad \bar{r} = x\hat{e}_x + z\hat{e}_z = (v_0 t\cos\theta, 0, v_0 t\sin\theta - \tfrac{1}{2}gt^2)$$

$$\rightarrow \bar{p} = m\dot{\bar{r}} = m(v_0\cos\theta, 0, v_0\sin\theta - gt)$$

$$\bar{L} = \bar{r}\times\bar{p} = (v_0 t\cos\theta, 0, v_0 t\sin\theta - \tfrac{1}{2}gt^2)\, m \times (v_0\cos\theta, 0, v_0\sin\theta - gt)$$

$$= -\hat{e}_y\left\{\cancel{v_0^2 t\cos\theta\sin\theta} - v_0 t\cos\theta\, gt - \cancel{v_0^2 t\sin\theta\cos\theta} + v_0\cos\theta\,\tfrac{1}{2}gt^2\right\} m$$

$$= \hat{e}_y \tfrac{1}{2} m g v_0 t^2 \cos\theta$$

$$\rightarrow \dot{\bar{L}} = \hat{e}_y\, m g v_0 t\cos\theta$$

— — — — — — —

Kraftverkun á ögnina í kasthreyfingu er

$$\bar{F} = -mg\hat{e}_z$$

$$\rightarrow \bar{N} = \bar{r}\times\bar{F} = (v_0 t\cos\theta, 0, v_0 t\sin\theta - \tfrac{1}{2}gt^2) \times (0, 0, -mg)$$

$$= \hat{e}_y\left\{+ m g v_0 t\cos\theta\right\}$$

$$\rightarrow \dot{\bar{L}} = \frac{d\bar{L}}{dt} = \bar{N}$$

(6) Fall í föstu þyngdarsviði með viðnámskr. $\sim v^4$

Setjum $v_0 = 0$

z

Hreyfijafna $\quad m\ddot{z} = -m\gamma v^4 + mg$

Líka og í dæmi (1) $\quad m v\dfrac{dv}{dz} = -m\gamma v^4 + mg$

$$\rightarrow \frac{v\,dv}{-\gamma v^4 + g} = dz \quad \text{eða} \quad \frac{v\,dv}{\gamma v^4 - g} = -dz$$

Notum GR 2.132.2

$$\int \frac{x\,dx}{bx^4 + a} = \frac{1}{4i\sqrt{ab}} \ln\left\{\frac{a + x^2 i\sqrt{ab}}{a - x^2 i\sqrt{ab}}\right\} \qquad ab < 0$$

$$\int_0^z dz' = \int_0^v \frac{v'dv'}{-\gamma(v')^4 + g} \rightarrow z = \frac{1}{4\sqrt{\gamma g}} \ln\left\{\frac{g + v^2\sqrt{\gamma g}}{g - v^2\sqrt{\gamma g}}\right\}$$

(19)

Án viðnáms gildir

$$\ddot{z} = g \quad \rightarrow \quad \dot{z} = v = gt$$

og

$$\frac{v\,dv}{g} = dz \quad \rightarrow \quad \int_0^v v'dv' = g\int_0^z dz'$$

$$\rightarrow \quad \frac{1}{2}v^2 = gz \quad \rightarrow \quad z = \frac{1}{2}\frac{v^2}{g}$$

$$z = \frac{1}{4\sqrt{rg}} \ln\left\{\frac{1 + v^2\sqrt{\frac{r}{g}}}{1 - v^2\sqrt{\frac{r}{g}}}\right\} \simeq \frac{1}{4\sqrt{rg}}\left[2v^2\sqrt{\frac{r}{g}} + \frac{2}{3}\left(\frac{r}{g}\right)^{3/2} v^6 + \dots\right.$$

$$\simeq \frac{v^2}{2g} + \frac{v^6}{6}\frac{r}{g^2} + \dots\dots$$

(20)

$$F = m\frac{dv}{dt} = mg - m r v^4$$

$$\rightarrow \quad \frac{dv}{-rv^4 + g} = dt$$

Notum GR 2.132.1

$$\int \frac{dx}{bx^4 + a} = \frac{\alpha'}{4a}\left[\ln\left(\frac{x+\alpha'}{x-\alpha'}\right) + 2\arctan\left(\frac{x}{\alpha'}\right)\right.$$

$$\alpha' = \sqrt[4]{-\frac{a}{b}}$$

því er hægt að leiða út

$$t = F(v)$$