Dæmi 1 Ögn með massa m í mættinu $U(x)=U_0\exp(-kx)$ hreyfist á bilinu $0 \leq x < \infty$, $x(0)=0$, $x(\tau)=x_0$

Finnum τ

$$dt = \pm \frac{dx}{\sqrt{\frac{2}{m}(E-U(x))}} = \pm \frac{dx}{\sqrt{\frac{2E}{m} - \frac{2U_0}{m}e^{-kx}}}$$

$$= \pm \frac{dx}{\sqrt{\frac{2U_0}{m}\left(\frac{E}{U_0} - e^{-kx}\right)}}$$

orkan fyrir kyrrstæða ögn í punktinum x=0 er

$$E = 0 + U(0) = U_0 \rightarrow dt = \pm\frac{dx}{\sqrt{\frac{2U_0}{m}(1-e^{-kx})}}$$

Skiptum um breytu

$$u = e^{-kx} \rightarrow du = -ke^{-kx}dx = -ku\,dx$$

$$\rightarrow dx = \frac{du}{-ku}$$

$$\tau = \int_0^{x_0}\frac{dx}{\sqrt{\frac{2U_0}{m}(1-e^{-kx})}} = -\frac{1}{k}\sqrt{\frac{m}{2U_0}}\int_1^{e^{-kx_0}}\frac{du}{u\sqrt{1-u}}$$

þetta heildi má finna á fylgiblöðum prófsins (E10.c)

$$\tau = \frac{1}{k}\sqrt{\frac{m}{2U_0}}\left[\ln\left\{\frac{2}{u}\sqrt{1-u}+\frac{2}{u}-1\right\}\right]_1^{e^{-kx_0}} = \frac{1}{k}\sqrt{\frac{m}{2U_0}}\left\{\ln\left\{2e^{kx_0}\sqrt{1-e^{-kx_0}} + 2e^{kx_0}-1\right\}\right\}$$

Svarið er því

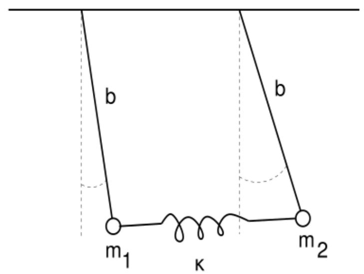
$$\tau = \frac{1}{k}\sqrt{\frac{m}{2U_0}}\left\{\ln\left[2e^{kx_0}\left(\sqrt{1-e^{-kx_0}}+1\right)-1\right]\right\}$$

Ágætt er að kanna að $\lim_{x_0\to 0}\tau(x_0) = 0$

eins og á að vera, og afellulausnin fyrir stórt kx_0 er, $kx_0 \gg 1$

$$\tau \xrightarrow[kx_0\gg 1]{} \frac{4kx_0}{k}\sqrt{\frac{m}{2U_0}} = 4x_0\sqrt{\frac{m}{2U_0}}$$

Dæmi 2



a) Án nálgunar

$$T = \frac{m_1}{2}(b\dot{\theta}_1)^2 + \frac{m_2}{2}(b_2\dot{\theta}_2)^2$$

$$U = m_1 g b(1-\cos\theta_1) + m_2 g(1-\cos\theta_2) + \frac{K}{2}\{b\sin\theta_1 - b\sin\theta_2\}^2$$

b) Gerum nálgun fyrir smá horn

$\sin\theta \simeq \theta$, $\cos\theta \simeq 1-\frac{\theta^2}{2}$

$$U = \frac{gb}{2}\{m_1\theta_1^2 + m_2\theta_2^2\} - \frac{Kb^2}{2}(\theta_1-\theta_2)^2$$

og hreyfiorkan er óbreytt

c) því verða

$$M = \begin{pmatrix} m_1 b^2 & 0 \\ 0 & m_2 b^2 \end{pmatrix}, \quad A = \begin{pmatrix} m_1 g b + K b^2 & -K b^2 \\ -K b^2 & m_2 g b + K b^2 \end{pmatrix}$$

Eigingildisverkefnið er

$$A\bar{a} = \omega^2 M \bar{a}$$

margföldun með M^{-1} eyðileggur samhverfuna

því er einfaldast að krefjast að krefjast

$$\det\left\{A - \omega^2 M\right\} = 0$$

sem verður

$$\begin{vmatrix} m_1 g b + K b^2 - \omega^2 m_1 b^2 & -K b^2 \\ -K b^2 & m_2 g b + K b^2 - \omega^2 m_2 b^2 \end{vmatrix} = 0$$

Ákveðan verður því

$$m_1 m_2 b^2\left[(g+\alpha_1-\omega^2 b)(g+\alpha_2-\omega^2 b)\right] - K^2 b^4 = 0 \qquad \alpha_1 = \frac{Kb}{m_1}, \quad \alpha_2 = \frac{Kb}{m_2}$$

$$\rightarrow g^2 + \alpha_1\alpha_2 + \omega^4 b^2 - 2g\omega^2 b + g(\alpha_1+\alpha_2) - \omega^2(\alpha_1+\alpha_2)b - \frac{(Kb)^2}{m_1 m_2} = 0$$

$$b^2\omega^4 - \omega^2\left\{2g+\alpha_1+\alpha_2\right\}b + (g+\alpha_1)(g+\alpha_2) - \frac{(Kb)^2}{m_1 m_2} = 0$$

svo lausnin er

$$\boxed{\omega_\pm^2 = \frac{g}{b} + \frac{1}{2b}(\alpha_1+\alpha_2) \pm \frac{1}{2b^2}\sqrt{(2g+\alpha_1+\alpha_2)^2 b^2 - 4\left[(g+\alpha_1)(g+\alpha_2) - \frac{(Kb)^2}{m_1 m_2}\right]b^2}}$$

d) skoðum lausnina fyrir tíðnirnar þegar $m_1 = m_2$

$$\omega_\pm^2 = \frac{g}{b} + \frac{K}{m} \pm \frac{1}{2b^2}\sqrt{4b^2\left(g+\frac{Kb}{m}\right)^2 - 4b^2\left[\left(g+\frac{Kb}{m}\right)^2 - \frac{(Kb)^2}{m^2}\right]}$$

$$= \frac{g}{b} + \frac{K}{m} \pm \frac{1}{b^2}\sqrt{\frac{(Kb)^2}{m^2}} = \frac{g}{b} + \frac{K}{m} \pm \frac{K}{m}$$

$$\rightarrow \boxed{\omega_\pm^2 = \begin{cases} \dfrac{g}{b} + \dfrac{2K}{m} \\ \dfrac{g}{b} \end{cases}}$$

sem er þekkt lausn fyrir tvo eins pendúla

Dæmi 3 Stjarfhlutur, $I_3 > I_2 = I_1$

Klukkan t=0

$$\bar{\omega} = \omega_1 \hat{e}_1 + \lambda \hat{e}_2 + \mu \hat{e}_3, \quad \lambda, \mu \ll \omega_1$$

skoðum tímaþróun snúningsins með truflanareikningi, jöfnur Eulers eru

$$\underbrace{\begin{aligned} (I_2 - I_3)\lambda\mu - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\mu\omega_1 - I_2\dot{\lambda} &= 0 \\ (I_1 - I_2)\lambda\omega_1 - I_3\dot{\mu} &= 0 \end{aligned}}$$

$$\rightarrow \dot{\omega}_1 \simeq 0 \rightarrow \omega_1 = \text{fasti}$$

$$\dot{\lambda} = \left(\frac{I_3 - I_1}{I_3}\omega_1\right)\mu$$

$$\dot{\mu} = \left(\frac{I_1 - I_2}{I_3}\omega_1\right)\lambda \quad \rightarrow \dot{\mu} = 0 \quad \rightarrow \mu = \text{fasti}$$

9

því fæst

$$\dot{\lambda} = \underbrace{\left(\frac{I_3 - I_1}{I_2}\,\omega_1\right)\mu}_{\text{fasti}} = \text{fasti}$$

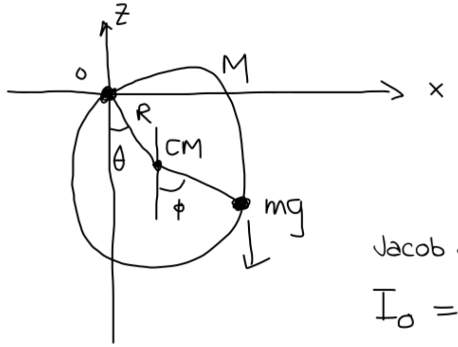
$$\longrightarrow \boxed{\lambda(t) = At + B},$$ þar sem A og B eru fastar

Samkvæmt þessu truflanamati vex því λ (eða ω_2) línulega og hættir að vera smátt miðað við ω_1. Þetta er truflanalausn og sýnir að snúningurinn er í upphafi óstöðugur. Vissulega mun ω_1 minnka og orkan haldast föst þegar nákvæmari lausn er skoðuð.

10

Dæmi 4 Gjörð og frjáls perla

a) Hnitin eru



$$x = R\{\sin\theta + \sin\phi\}$$

$$y = -R\{\cos\theta + \cos\phi\}$$

Jacob Steiner:

$$I_o = I_{cm} + MR^2 = 2MR^2$$

$$U = U_{arc} + U_m = \{-MgR\cos\theta\} + \{-mgR\cos\theta - mgR\cos\phi\}$$

$$= -MgR\cos\theta - mgR\{\cos\theta + \cos\phi\}$$

11

$$T = \underbrace{\frac{I_o\dot{\theta}^2}{2}}_{\text{gjörð}} + \underbrace{\frac{m}{2}\{\dot{x}^2 + \dot{y}^2\}}_{\text{perla}}$$

$$\dot{x} = R\{\dot{\theta}\cos\theta + \dot{\phi}\cos\phi\}$$

$$\dot{y} = R\{\dot{\theta}\sin\theta + \dot{\phi}\sin\phi\}$$

$$T = MR^2\dot{\theta}^2 + \frac{mR^2}{2}\left\{(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi)^2 + (\dot{\theta}\sin\theta + \dot{\phi}\sin\phi)^2\right\}$$

$$\simeq MR^2\dot{\theta}^2 + \frac{mR^2}{2}\{\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2\}$$

$$U \simeq \frac{MgR}{2}\theta^2 + \frac{mgR}{2}\{\theta^2 + \phi^2\}$$

í lægstu nálgun fyrir hornin

12

b) Tvö alhnit, ϕ og θ

$$L \simeq M(R\dot{\theta})^2 + \frac{mR^2}{2}\{\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2\} - \frac{MgR}{2}\theta^2 - \frac{mgR}{2}\{\theta^2 + \phi^2\}$$

Hreyfijöfnurnar eru því

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$-MgR\theta - mgR\theta - \frac{d}{dt}\{2MR^2\dot{\theta} + mR^2\dot{\theta} + mR^2\dot{\phi}\} = 0$$

$$-mgR\phi - \frac{d}{dt}\{mR^2\dot{\theta} + mR^2\dot{\phi}\} = 0$$

Þær má því skrifa sem

$$(2M+m)\ddot\theta + m\ddot\phi + \frac{g(M+m)}{R}\theta = 0$$

$$\ddot\theta + \ddot\phi + \frac{g}{R}\phi = 0$$

c) Alskriðþungar

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = 2M^2\dot\theta + mR^2\{\dot\theta + \dot\phi\}$$

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = mR^2\{\dot\phi + \dot\theta\}$$

d) Fall Hamiltons

$$H = p_\theta\dot\theta + p_\phi\dot\phi - L$$

$$= 2M(R\dot\theta)^2 + m(R\dot\theta)^2 + mR^2\dot\theta\dot\phi + mR^2\dot\phi^2 + mR^2\dot\theta\dot\phi$$

$$-M(R\dot\theta)^2 - \frac{mR^2}{2}\{\dot\theta^2 + 2\dot\theta\dot\phi + \dot\phi^2\} + \frac{MgR}{2}\theta^2 + \frac{mgR}{2}\{\theta^2 + \phi^2\}$$

$$= M(R\dot\theta)^2 + \frac{mR^2}{2}\{\dot\theta^2 + 2\dot\theta\dot\phi + \dot\phi^2\} + \frac{MgR}{2}\theta^2 + \frac{mgR}{2}\{\theta^2 + \phi^2\}$$

$$= \frac{(p_\theta - p_\phi)^2}{MR^2} + \frac{p_\phi^2}{2mR^2} + \frac{gR}{2}\{(M+m)\theta^2 + m\phi^2\}$$

Kórjöfnur Hamiltons

$$\dot q = \frac{\partial H}{\partial p}, \qquad -\dot p = \frac{\partial H}{\partial q}$$

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{2(p_\theta - p_\phi)}{MR^2}$$

$$-\dot p_\theta = \frac{\partial H}{\partial\theta} = 2R\theta(m+M)$$

$$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{2}{MR^2}(p_\phi - p_\theta) + \frac{p_\phi}{mR^2}$$

$$-\dot p_\phi = \frac{\partial H}{\partial\phi} = gmR\phi$$

Dæmi 5 Miðlægt mætti (ekki háð horni) $U(r) = U_0\ln(r/r_0)$

U_0 og r_0 jákvæðir fastar

a) Virkamættið

$$V(r) = U(r) + \frac{\ell^2}{2mr^2}, \qquad \ell = mr^2\dot\theta$$

fasti, þar sem hornið kemur ekki í falli Lagrange

b) Pólhnit

$$L = T - U = \frac{m\dot r^2}{2} + \frac{m(r\dot\theta)^2}{2} - U_0\ln(r/r_0)$$

c) Hnitið θ er rásað þar sem það kemur ekki fyrir í $L(r,\dot r,\dot\theta)$

d) Finna geisla stöðugrar hringbrautar. Byrjum á hreyfijöfnunni f)

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot r}\right) = 0 \longrightarrow \boxed{\ddot r - \frac{\ell^2}{m^2r^3} + \frac{U_0}{mr} = 0}$$

Jafnvægisfjarlægð

$$\frac{\partial V}{\partial r} = \frac{U_0}{r_0} - \frac{\ell^2}{m r_0^3} = 0 \longrightarrow r_0^2 = \frac{\ell^2}{U_0 m}$$

$$\longrightarrow \boxed{r_0 = \frac{|\ell|}{\sqrt{U_0 m}}}$$

e) Greinilega eru ekki til efri mörk fyrir hverfiþunga agnarinnar

g) smáar sveiflur, nágun hreyfijöfnu

$$\ddot{r} - \frac{\ell^2}{m^2 r^3} + \frac{U_0}{m r} = 0, \qquad r = r_0 + \delta, \quad \frac{1}{(r_0+\delta)^n} \simeq \frac{1}{r_0^n}\left[1 - \frac{n\delta}{r_0}\right]$$

$$\longrightarrow \ddot{\delta} - \frac{\ell^2}{m^2 r_0^3}\left[1 - \frac{3\delta}{r_0}\right] + \frac{U_0}{m r_0}\left[1 - \frac{\delta}{r_0}\right] \simeq 0$$

Lágmarksskilyrðin

Því stendur eftir hreyfijafnan fyrir smáar sveiflur

$$\ddot{\delta} + \underbrace{\left[\frac{3\ell^2}{m^2 r_0^4} - \frac{U_0}{m r_0^2}\right]}\delta = 0$$

$$\frac{3\ell^2 U_0^2 m^2}{m^2 \ell^4} - \frac{U_0^2 m}{m \ell^2} = 3\frac{U_0^2}{\ell^2} - \frac{U_0^2}{\ell^2} = \frac{2U_0^2}{\ell^2}$$

$$\longrightarrow \boxed{\omega = \sqrt{2}\left|\frac{U_0}{\ell}\right|}$$

h) U(r) er rétta þyngdarmætti punktuppsprettu í tvívídd. Ögn um hana er alltaf bundin, hver sem orka hennar eða skriðþungi er. Þetta er á allt annan hátt í þrívídd eins og við höfum skoðað.