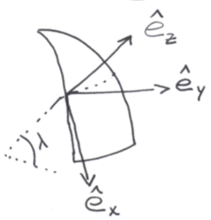
3

Norðurhvel, ögn hent upp í hæð h yfir punkti á yfirborði jarðar sýna hvar hún lendir (hnikað til vesturs)

sleppa loftmótstöðu, $\frac{h}{a_J} << 1$

Hnit eins og á mynd 10-9 í bók



Hröðun vegna krafts Coriolis er

$$\bar a = -2\bar\omega\times\dot{\bar r}$$

$$\bar\omega\times\dot{\bar r} = \begin{vmatrix} \bar i & \bar j & \bar k \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & 0 & \dot z \end{vmatrix} = (0, \omega\cos\lambda\cdot\dot z, 0)$$

því nálgun!

$$\to \bar a = -2\bar\omega\times\dot{\bar r} = -2\omega\dot z\cos\lambda\,\hat e_y$$

$$\to \ddot y = -2\omega\dot z\cos\lambda$$

Fyrir z-stefnu

$$z = v_0 t - \frac{gt^2}{2}$$

$$v_z^2 = v_0^2 - 2zg$$

$$\Rightarrow v_0 = \sqrt{2gh}$$

heildum í tíma

$$\dot y = -2\omega z\cos\lambda + C_1$$

$$\dot y(0) = 0 \to C_1 = 0$$

$$\dot y = -2\omega\left\{v_0 t - \frac{gt^2}{2}\right\}\cos\lambda$$

$$= -\omega\cos\lambda\cdot\left\{2v_0 t - gt^2\right\}$$

heildum aftur

$$y = -\omega\cos\lambda\cdot\left\{v_0 t^2 - \frac{gt^3}{3}\right\} + C_2$$

$$y(0) = 0 \to C_2 = 0$$

$$y = -\omega\cos\lambda\cdot\left\{v_0 t^2 - \frac{gt^3}{3}\right\}$$

flugtími: $T = \frac{2v_0}{g}$

$$\to y = -\omega\cos\lambda\cdot\left\{v_0\left(\frac{2v_0}{g}\right)^2 - \frac{g}{3}\left(\frac{2v_0}{g}\right)^3\right\}$$

$$= -\omega\cos\lambda\cdot\frac{v_0^3}{g^2}\cdot\left\{4 - \frac{8}{3}\right\}$$

$$= -\omega\cos\lambda\cdot\frac{4}{3}\frac{v_0^3}{g^2}$$

$$v_0 = \sqrt{2gh}$$

$$\to y = -\omega\cos\lambda\cdot\frac{4}{3g^2}(2gh)^{3/2}$$

$$= -\frac{4\omega}{3}\cos\lambda\cdot\sqrt{\frac{8h^3}{g}}$$

í vesturátt

5

Skoðum hreyfingu í krafti

$$F(r) = -\frac{k}{r^3} \to U(r) = -\frac{k}{2r^2}$$

Virkamættið

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2}$$

$$= \frac{1}{2}\left\{\frac{\ell^2}{\mu} - k\right\}\frac{1}{r^2}$$

miðflóttakrafturinn og mættið hafa sama form

Schrödinge - - - -

Athugum hreyfijöfnuna

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2 u^2} F(1/u) = -\frac{\mu}{\ell^2 u^2}\{-ku^3\} = +\frac{\mu k}{\ell^2} u$$

$$\rightarrow \boxed{\frac{d^2u}{d\theta^2} + \left\{1 - \frac{\mu k}{\ell^2}\right\} u = 0}$$

skoðum tilfellin sem koma til greina

① $\ell^2 = \mu k$ (ℓ tengist hverfiþunga)

$$\rightarrow \frac{d^2u}{d\theta^2} = 0 \quad \text{með lausn} \quad u = A\theta + B$$

$$u = \frac{1}{r} \rightarrow r = \frac{1}{A\theta + B}$$

gormhreyfing að miðju

② $\ell^2 > \mu k \rightarrow \left\{1 - \frac{\mu k}{\ell^2}\right\} = \gamma^2 > 0$

$$\rightarrow \boxed{\frac{d^2u}{d\theta^2} + \gamma^2 u = 0}$$

með lausn $u = A\cos(\gamma\theta + \delta) \rightarrow r = \frac{1}{A\cos(\gamma\theta+\delta)}$

$u \in [-1,+1] \rightarrow Ar \in \mathbb{R}$ og brautin er opin

③ $\ell^2 < \mu k \rightarrow \left\{1 - \frac{\mu k}{\ell^2}\right\} = -\kappa^2$, með $\kappa^2 > 0$

$$\rightarrow \boxed{\frac{d^2u}{d\theta^2} - \kappa^2 u = 0} \rightarrow u = A\cosh(\kappa\theta + \delta)$$

$$r = \frac{1}{A\cosh(\kappa\theta + \delta)} \qquad \cosh x \in [1, \infty)$$

$\rightarrow Ar \in (0, 1]$

hreyfing að miðju

Stöðugleiki hringbrautar

$$g(r) = \frac{1}{\mu}\frac{\partial U}{\partial r} = \frac{k}{\mu r^3}, \qquad r \rightarrow \rho + x \quad \text{þ.s.} \quad x \ll \rho$$

frávik frá hring

$$\ddot{x} - \frac{\ell^2}{\mu^2\rho^3\left[1 + \frac{x}{\rho}\right]^3} = -g(\rho + x)$$

$$= -\frac{k}{\mu\rho^3\left[1 + \frac{x}{\rho}\right]^3}$$

$$\rightarrow \boxed{\ddot{x} + \left\{k - \frac{\ell^2}{\mu}\right\}\frac{1}{\mu\rho^3\left(1 + \frac{x}{\rho}\right)^3} = 0}$$

$$g(\rho) = \frac{\ell^2}{\mu^2\rho^3} \rightarrow k = \frac{\ell^2}{\mu}$$

$$\ddot{r}\big|_{r=\rho} = 0 \quad \rightarrow \quad k = \frac{\ell^2}{\mu}$$ fyrir stöðuga hringbraut

$\rightarrow \ddot{x} = 0 \leftarrow$ en þessi jafna gefur ekki takmarkaða lausn $\rightarrow$ engin stöðug hringbraut

Dæmi 6

Ögn ferðast eftir einvíðum vökva þar sem aðeins viðnámskraftur $F = mk\sqrt{v}$ verkar gegn hreyfingunni.

Finna $v(t)$ ef $v(0) = v_0$

Hreyfijafnan er

$$m\ddot{x} = -mk\sqrt{v}$$

$$\frac{d^2x}{dt^2} = -k\sqrt{\left(\frac{dx}{dt}\right)} \rightarrow \frac{dv}{dt} = -k\sqrt{v}$$

Heildum

$$\boxed{\frac{dv}{\sqrt{v}} = -k\,dt}$$

$$\int_{v_0}^{v(t)} \frac{du}{\sqrt{u}} = -k\int_0^t d\tau \longrightarrow 2\left\{\sqrt{v(t)} - \sqrt{v_0}\right\} = -kt$$

eða $$\sqrt{v(t)} - \sqrt{v_0} = -\frac{kt}{2}$$

og því $$\sqrt{v(t)} = \sqrt{v_0} - \frac{kt}{2}$$ ef $\sqrt{v_0} - \frac{kt}{2} \geq 0$

og $$\boxed{v(t) = \left(\sqrt{v_0} - \frac{kt}{2}\right)^2}$$

Finna $x(t)$

$$\frac{dx}{dt} = \left(\sqrt{v_0} - \frac{kt}{2}\right)^2$$

Heildum

$$dx = \left(\sqrt{v_0} - \frac{kt}{2}\right)^2 dt$$

$$\longrightarrow \int_{x_0}^{x(t)} du = \int_0^t d\tau \left(\sqrt{v_0} - \frac{kt}{2}\right)^2$$ ef $\sqrt{v_0} - \frac{kt}{2} \geq 0$

$$\boxed{x(t) - x_0 = t v_0 - \frac{k}{2}\sqrt{v_0}\, t^2 + \left(\frac{k}{2}\right)^2 \frac{t^3}{3}}$$

Ögnin stöðvast þegar $v(t) = 0$, þ.e. þegar

$$\sqrt{v_0} - \frac{kt}{2} = 0 \longrightarrow \frac{kt}{2} = \sqrt{v_0}$$

eða $t = \frac{2}{k}\sqrt{v_0}$, setjum $t_s = \frac{2}{k}\sqrt{v_0}$

þá er fjarlægðin sem ögnin fer $x(t_s)$

$$\boxed{x(t_s) = \frac{2}{k} v_0^{3/2} - \frac{2}{k} v_0^{3/2} + \left(\frac{k}{2}\right)^2 \frac{1}{3}\left(\frac{2}{k}\right)^3 v_0^{3/2} = \frac{1}{3}\,\frac{2}{k}\, v_0^{3/2}}$$

Skoðum lausnir

$$v(t) = \left(\sqrt{v_0} - \frac{kt}{2}\right)^2 = v_0\left(1 - \frac{k}{2\sqrt{v_0}}\, t\right)^2$$

ef $\frac{k}{2\sqrt{v_0}}\, t \ll 1$ þá fæst

$$v(t) \approx v_0\left\{1 - \frac{k}{\sqrt{v_0}}\, t + \cdots\right\} = v_0 - k\sqrt{v_0}\, t + \cdots$$ (*)

og

$$x(t) - x_0 = t v_0 - \frac{k}{2}\sqrt{v_0}\, t^2$$ (*)

því er eins og í upphafi sé hröðun $\left(\frac{k\sqrt{v_0}}{1}\right)$ á móti ögninni, eins og búast mætti við dæmi 7

b í jafnvægi

Alhnit geta verið θ og tímaháð lengd pendúls l

$$T = \frac{m}{2}\left\{\dot{l}^2 + (l\dot{\theta})^2\right\}$$

$$U = mgz + \frac{1}{2}k(l-b)^2 = -mgl\cos\theta + \frac{1}{2}k(l-b)^2$$

Lagrange jöfnurnar eru því

$$\frac{\partial L}{\partial l} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) = 0 \quad \rightarrow \quad ml\dot{\theta}^2 + mg\cos\theta - k(l-b) - m\ddot{l} = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \quad \rightarrow \quad -mgl\sin\theta - \frac{d}{dt}\left\{2l^2\dot{\theta}\right\}\frac{m}{2} = 0$$

$$\rightarrow \boxed{\begin{aligned} &\ddot{l} - l\dot{\theta}^2 - g\cos\theta + \frac{k}{m}(l-b) = 0 \\ &\ddot{\theta} + \frac{2\dot{l}}{l}\dot{\theta} + \frac{g}{l}\sin\theta = 0 \end{aligned}}$$