# Hreyfing snúðs án ytri krafts, samhverfur snúður



samhverfur $I_1 = I_2 \neq I_3$

Í fastakerfinu er CM kyrr eða á jafnri hreyfingu

setjum CM í O-ið á fasta kerfinu

L er fasti í fasta kerfinu

þ.s. engir kraftar verka á snúðinn

Jöfnur Eulers verða í kerfi hlutar

$$\boxed{\begin{aligned} (I_1 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\omega_3\omega_1 - I_1\dot{\omega}_2 &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}}$$

Notum höfuðása til að skilgreina hnitakerfi hlutar

Gerum ráð fyrir að $\vec{\omega}$ liggi ekki eftir höfuðás snúðs

$\omega_3(t) = $ fasti

veljum $\Omega \equiv \dfrac{I_3 - I_1}{I_1}\omega_3$ fasti

þá fæst

$$\begin{cases} \dot{\omega}_1 + \Omega\omega_2 = 0 \\ \dot{\omega}_2 - \Omega\omega_1 = 0 \end{cases}$$

leggjum hreyfijöfnumar saman

$$(\dot{\omega}_1 + i\dot{\omega}_2) - i\Omega(\omega_1 + i\omega_2) = 0$$

Skilgreinum

$$\eta \equiv \omega_1 + i\omega_2$$

$$\dot{\eta} - i\Omega\eta = 0$$

með lausn

$$\eta(t) = Ae^{i\Omega t}$$

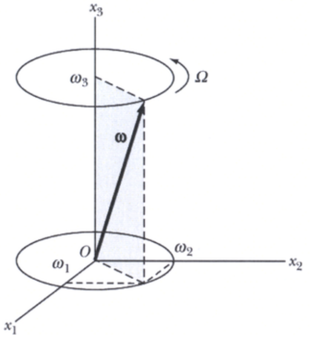
$$\rightarrow \omega_1 + i\omega_2 = A\cos(\Omega t) + iA\sin(\Omega t)$$

$$\rightarrow \omega_1(t) = A\cos(\Omega t)$$

$$\omega_2(t) = A\sin(\Omega t)$$

$\omega_3 = \text{fasti}$

Hnitakerfi snúðs




$$|\omega| = \omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$$

$$= \sqrt{A^2 + \omega_3^2} = \text{fasti}$$

$$\boxed{\begin{aligned} |\omega| &= \text{fasti} \\ \omega_3 &= \text{fasti} \end{aligned}}$$

$x_3$-ás ← samhverfuás

$\bar{\omega}$ veltur um samhverfuás hluta

með hornferð $\Omega$, $\bar{\omega}$ teiknar keiluhlutar (body cone)

Enginn ytri kraftur

$\bar{L}$ : fasti í fasta kerfinu

CM er fast í O

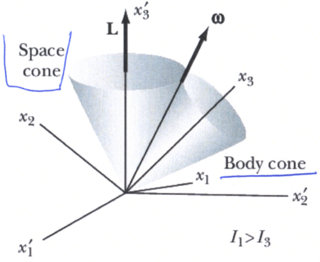
$T = \frac{1}{2}\underbrace{\bar{\omega}\cdot\bar{L}} = \text{fasti}$

$\bar{\omega}$ veltur um $\bar{L}$ með föstu horni

$\bar{L}$, $\bar{\omega}$, og $\underbrace{x_3\text{-ásinn}}_{\text{samhverfuás hlutar}}$ liggja í sömu sléttu

Athugum

$$\bar{\omega} \times \hat{e}_3 = \omega_2\hat{e}_1 - \omega_1\hat{e}_2$$




$$\bar{L}\cdot(\bar{\omega}\times\hat{e}_3) = \bar{L}\cdot(\omega_2\hat{e}_1 - \omega_1\hat{e}_2)$$
$$= L_1\omega_2 - \omega_1 L_2$$
$$= I_1\omega_1\omega_2 - I_2\omega_1\omega_2$$
$$= 0, \text{ því } I_1 = I_2$$

$\boxed{\bar{L} \text{ er í sléttu } \bar{\omega} \text{ og } \hat{e}_3}$

# Dæmi

Skoðum fyrir langan snúð $I_1 > I_3$ eða flatan $I_3 > I_1$

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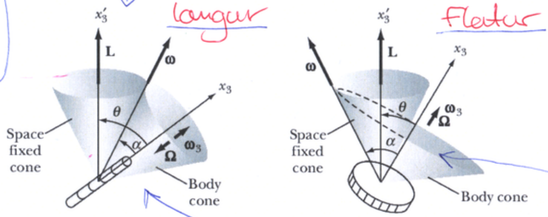


Prolate, $I_1 > I_3$
$\mathbf{\Omega}$, $\boldsymbol{\omega}_3$ have opposite signs.
(a)

Oblate, $I_3 > I_1$
$\mathbf{\Omega}$, $\boldsymbol{\omega}_3$ have same sign.
(b)

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3$$

$\bar{L} \parallel x_3'$-ás

þá er Euler hornið $\theta$ hornið milli $\bar{L}$ og $x_3$

Á vissum tímapunkti er $\hat{e}_2$ í sléttu $\bar{L}, \bar{\omega}$ og $\hat{e}_3$

$$\rightarrow \begin{cases} L_1 = 0 \\ L_2 = L\sin\theta \\ L_3 = L\cos\theta \end{cases}$$

Ef hornið milli $\bar{\omega}$ og $x_3$-áss er $\alpha$

$$\begin{cases} \omega_1 = 0 \\ \omega_2 = \omega\sin\alpha \\ \omega_3 = \omega\cos\alpha \end{cases}$$

$$L_i = I_i\omega_i$$

$$\begin{cases} L_1 = I_1\omega_1 = 0 \\ L_2 = I_1\omega_2 = I_1\omega\sin\alpha \\ L_3 = I_3\omega_3 = I_3\omega\cos\alpha \end{cases}$$

$$\frac{L_2}{L_3} = \tan\theta = \frac{I_1}{I_3}\tan\alpha$$

Langur : $I_1 > I_3$

$\rightarrow \theta > \alpha$

flatur : $I_3 > I_1$

$\rightarrow \alpha > \theta$

$\bar{L}$ er fasti → rúmkeilan er föst

Keila snúðs veltur innan ~~eða~~ utan á rúm keilunni

Snertilínan er standar snúningsás hreyfingarinnar

↑ í henni liggur $\bar{\omega}$ sem stilgreinir keilurnar

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Dæmi  Með hvaða hornhraða snúast $x_3$ og $\bar{\omega}$ um $\bar{L}$?

$\hat{e}_3, \bar{\omega}$ og $\bar{L}$ í sömu sléttu → $\hat{e}_3$ og $\bar{\omega}$ hafa sama hraða um L

$\dot{\phi}$ er hornhraðinn um $x_3'$ á sama vissa tíma og hér á undan

$(\hat{e}_2$ í sléttu $\hat{e}_3, \bar{\omega}$ og $\bar{L})$ → $\psi = 0$ → $\omega_2 = \dot{\phi}\sin\theta$ → $\dot{\phi} = \frac{\omega_2}{\sin\theta}$

hér að undan sékkst $\omega_2 = \omega\sin\alpha$ → $\dot{\phi} = \frac{\omega\sin\alpha}{\sin\theta}$

$$\dot{\phi} = \frac{\omega\sin\alpha}{\sin\theta} = \omega\frac{L_2 L}{I_1\omega L_2} = \frac{L}{I_1}$$

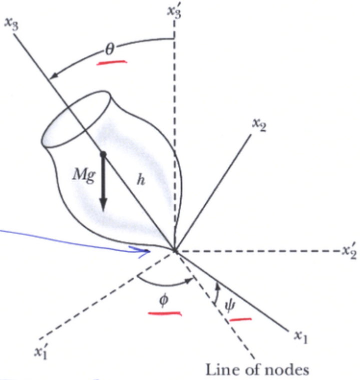
fra síðu 4

# Samhverfur snúður með fastan punkt

(Lagrange)

Heppilegt að nota „fasta" oddinn sem upphaf beggja hnitakerfa

→ Euler hornin eru mjög þægileg að nota




$I_1 = I_2$, gerum ráð fyrir $I_3 \neq I_1$

$$\boxed{T = \frac{1}{2}\sum_i I_i \omega_i^2 = \frac{I_1}{2}\left\{\omega_1^2 + \omega_2^2\right\} + \frac{I_3}{2}\omega_3^2}$$

En áður var leitt út

$$\left.\begin{aligned}
\omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\
\omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\
\omega_3 &= \dot{\phi}\cos\theta + \dot{\psi}
\end{aligned}\right\} \longrightarrow$$

$$\omega_1^2 + \omega_2^2 = \dot{\phi}^2\sin^2\theta + \dot{\theta}^2$$

$$\omega_3^2 = \left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$$

og því

$$T = \frac{I_1}{2}\left\{\dot\phi^2 \sin^2\theta + \dot\theta^2\right\} + \frac{I_3}{2}\left\{\dot\phi\cos\theta + \dot\psi\right\}^2$$

$$L = T - Mgh\cos\theta$$

L er (cyclic) óháður $\phi$ og $\psi$, því eru korsknúþungar þessara breyta varðveittar stærðir

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = \left\{I_1 \sin^2\theta + I_3\cos^2\theta\right\}\dot\phi + I_3\dot\psi\cos\theta = \text{fasti} \quad (*)$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3\left\{\dot\psi + \dot\phi\cos\theta\right\} = \text{fasti} \quad (**)$$

sem eru horn $\rightarrow$ $p_\phi$ og $p_\psi$ eru hverfiþungar

Hverfiþungarnir um $x_3'$ og $x_3$-ás eru fastir

Við notum varðveislu hverfiþunganna (*) og (**) til að leysa eftir við $\dot{\phi}$ og $\dot{\psi}$

$$(**) \longrightarrow \dot{\psi} = \frac{P_\psi - I_3\dot{\phi}\cos\theta}{I_3}$$

notum í (*)

$$\longrightarrow \left\{I_1\sin^2\theta + I_3\cos^2\theta\right\}\dot{\phi} + \left\{P_\psi - I_3\dot{\phi}\cos\theta\right\}\cos\theta = P_\phi$$

$$\longrightarrow \quad I_1\sin^2\theta\cdot\dot{\phi} + P_\psi\cos\theta = P_\phi$$

$$\longrightarrow \quad \dot{\phi} = \frac{P_\phi - P_\psi\cos\theta}{I_1\sin^2\theta}$$

$$\longrightarrow \dot{\psi} = \frac{P_\psi}{I_3} - \frac{\left\{P_\phi - P_\psi\cos\theta\right\}\cos\theta}{I_1\sin^2\theta}$$

Kerfið er geymið,

$$\rightarrow E = \frac{I_1}{2}\left\{\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right\} + \frac{I_3}{2}\omega_3^2 + Mgh\cos\theta = \text{fasti}$$

berum saman

$$P_\psi = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = \text{fasti}$$

og $\quad \omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$

Gefur $\quad P_\psi = I_3\omega_3 = \text{fasti}$ eða $I_3\omega_3^2 = \dfrac{P_\psi^2}{I_3} = \text{fasti}$

því er $E - \dfrac{I_3}{2}\omega_3^2 = E'$ <u>varðveitt stærð</u>

$$E' \equiv E - \frac{I_3}{2}\omega_3^2 = \frac{I_1}{2}\left\{\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right\} + Mgh\cos\theta = \text{fasti}$$

$$\boxed{E' = \frac{I_1\dot{\theta}^2}{2} + \frac{\{P_\phi - P_\psi\cos\theta\}^2}{2I_1\sin^2\theta} + Mgh\cos\theta}$$

Við getum skilgreint virkt mætti (effective potential)

$$\left\{ E' = \frac{I_1 \dot{\theta}^2}{2} + V(\theta) \right.$$

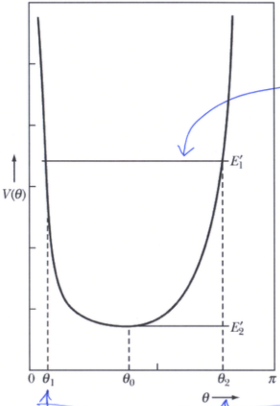
með

$$V(\theta) \equiv \frac{\{P_\phi - P_\psi \cos\theta\}^2}{2 I_1 \sin^2\theta} + Mgh\cos\theta$$

$$\rightarrow \quad \dot{\theta}^2 = \frac{2}{I_1}\left(E' - V(\theta)\right)$$

$$\rightarrow \quad \frac{d\theta}{dt} = \sqrt{\frac{2}{I_1}\left(E' - V(\theta)\right)}$$

$$\rightarrow \quad t(\theta) = \int \frac{d\theta}{\sqrt{\frac{2}{I_1}\left(E' - V(\theta)\right)}}$$




hreyfing með $E_1'$ er takmörkuð á þessu svæði

↓

Viðsnúningspunktar

gefur formlega nákvæma lausn en við getum lært gríðarlegt með athugunum á hreyfijöfnunum

* Halli snúðs með $E_1'$ takmarkast við bilið $\theta_1 \leq \theta \leq \theta_2$

* Fyrir $E' = E_2' = V_{min}$ er aðeins eitthvert mögulegt $\theta_0$
og snúðurinn er í stöðugri veltu

↳ Hægt er að finna $\theta_0$

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = \frac{-\cos\theta_0\left\{P_\phi - P_\psi \cos\theta_0\right\}^2 + P_\psi \sin^2\theta_0 \left\{P_\phi - P_\psi \cos\theta_0\right\}}{I_1 \sin^3\theta_0}$$

$$- Mgh \sin\theta_0 = 0$$

Ef $\beta \equiv P_\phi - P_\psi \cos\theta_0$

þá fæst

$$\cos\theta_0 \cdot \beta^2 - P_\psi \sin^2\theta_0 \cdot \beta + MghI_1 \sin^4\theta_0 = 0$$

$$\longrightarrow \beta = \frac{P_\psi \sin^2\theta_0}{2\cos\theta_0}\left\{1 \pm \sqrt{1 - \frac{4MghI_1\cos\theta_0}{P_\psi^2}}\right\}$$

$\beta \in \mathbb{R}$

$$\rightarrow p_\psi^2 \geq 4MghI_1 \cos\theta_0 \qquad \text{og áður } p_\psi = I_3\omega_3$$

$$\rightarrow \omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1\cos\theta_0}$$

stöðug velta er aðeins möguleg við fast horn $\theta_0$ ef spunahraðinn er nogilegur

Áður var komið

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$$

fyrir $\theta = \theta_0$ er þetta

$$\dot{\phi} = \frac{\beta}{I_1 \sin^2\theta_0}$$

því geta ræturnar fyrir $\beta_\pm$ tvenns konar veltu

$$\dot{\phi}_{0(+)} \rightarrow \text{hröð velta}$$

$$\dot{\phi}_{0(-)} \rightarrow \text{hæg velta}$$

Ef $\omega_3$ (eða $P_\psi$) sýnir hvaðan spuna, má velja

$$\dot\phi_{0(+)} \simeq \frac{I_3\omega}{I_1\cos\theta_0}, \qquad \underbrace{\dot\phi_{0(-)} \simeq \frac{Mgh}{I_3\omega_3}}_{\text{sést venjulega}}$$

Við höfum skoðað allt hér fyrir $\theta_0 < \pi/2$

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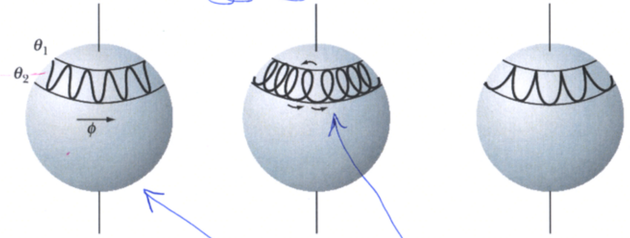
Ef $\theta_0 > \frac{\pi}{2}$ þá er staðan innan rótar í $\beta$ alltaf $>0$

$\rightarrow$ engin mörk á $\omega_3$ og hoga og hvaða veltan verða í sitthvora áttina

CM fyrir hvaðan fastan punkt

# Vagg (nutatión)

vagg og velta




(a) (b) (c)

$\theta_1 < \theta < \theta_2$

almenna tilfellið

$$\underbrace{\dot{\phi} = \frac{P_\phi - P_\psi \cos\theta}{I_1 \sin^2\theta}}$$

→ $\dot{\phi}$ getur skipt um formerki þegar $\theta$ fer milli markanna $\theta_1$ og $\theta_2$ (er háð gildum á $P_\phi$ og $P_\psi$)

Ef $\dot{\phi}$ skiptir ekki um formerki → stöðuvelta
en sveifla milli $\theta_1$ og $\theta_2$ → vagg

Ef $\dot{\phi}$ skiptir um formerki þá fær $\dot{\phi}$ mismunandi stefnu við $\theta_1$ og $\theta_2$
→ vagg með lykkjum

Ef hlutfall $P_\phi$ og $P_\psi$ er $(P_\phi - P_\psi \cos\theta)\big|_{\theta=\theta_1} = 0$ → $\dot{\phi}\big|_{\theta=\theta_1} = 0$ $\quad \dot{\theta}\big|_{\theta=\theta_1} = 0$
venjulega er snúði sleppt þannig ← tog niður vega þyngdar

## Tölulegt lausn fyrir snúð

Ekki er heppilegt að nota $E$ og $E'$. Í stað þess er hreyfijafnan út frá $L$ beint, notum Euler-Lagrange

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 0$$

$$\rightarrow \quad \ddot\theta = \frac{Mgh}{I_1}\sin\theta - \frac{I_3}{I_1}\left\{\dot\phi\cos\theta + \dot\psi\right\}\dot\phi\sin\theta + \dot\phi^2\sin\theta\cos\theta$$

auk

$$\dot\phi = \frac{S_z - B_3\cos\theta}{I_1\sin^2\theta}, \quad \dot\psi = \frac{B_3}{I_3} - \frac{(S_z - B_3\cos\theta)\cos\theta}{I_1\sin^2\theta}$$

þar sem

$$B_3 = P_\psi = \text{fasti}, \quad S_z = P_\phi = \text{fasti}$$

Veljum

$$I_3 = \frac{2}{5} M R^2 , \qquad M = 0{,}1 \text{ Kg}, \quad R = 0{,}04 \text{m},$$

$$I_2 = I_1 = \frac{I_3}{5}$$

$$h = R + \delta , \qquad \delta = 0{,}01 \text{ m}$$

Notum síðan forritið góða af heimasíðu námskeiðs til að reikna tíma þróun kerfisins þegar við getum

$\dot{\psi}(0)$

$\dot{\phi}(0)$

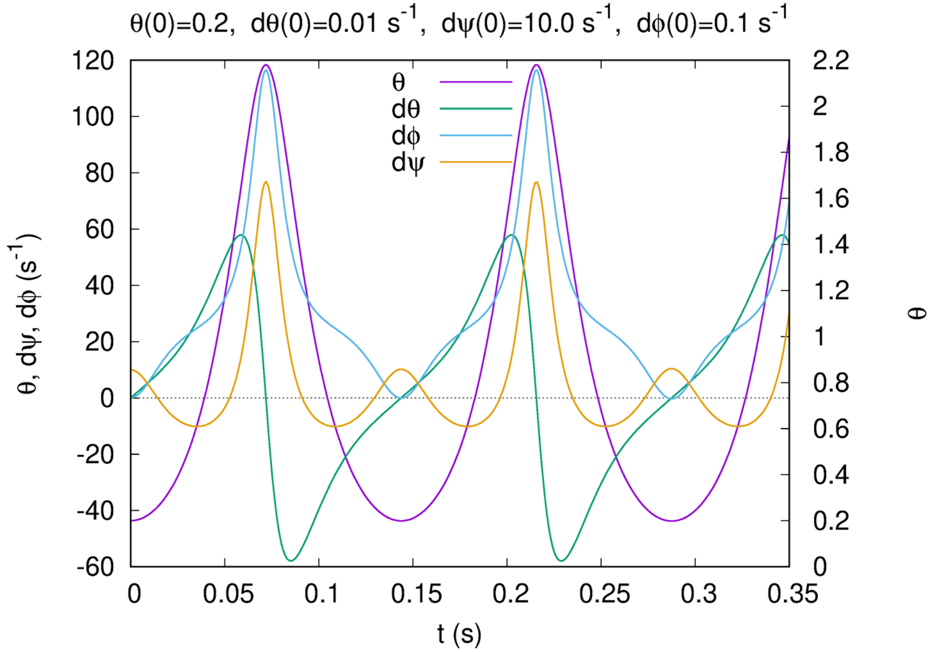
$\theta(0)$

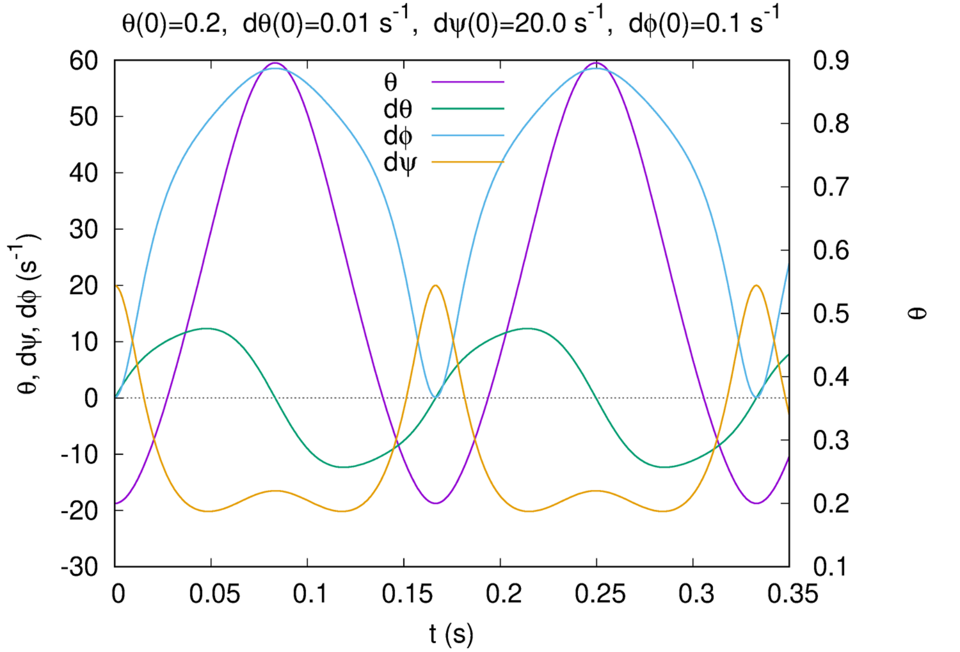
$\dot{\theta}(0)$

$\longrightarrow$ $B_3$ og $S_z$ eru fastir

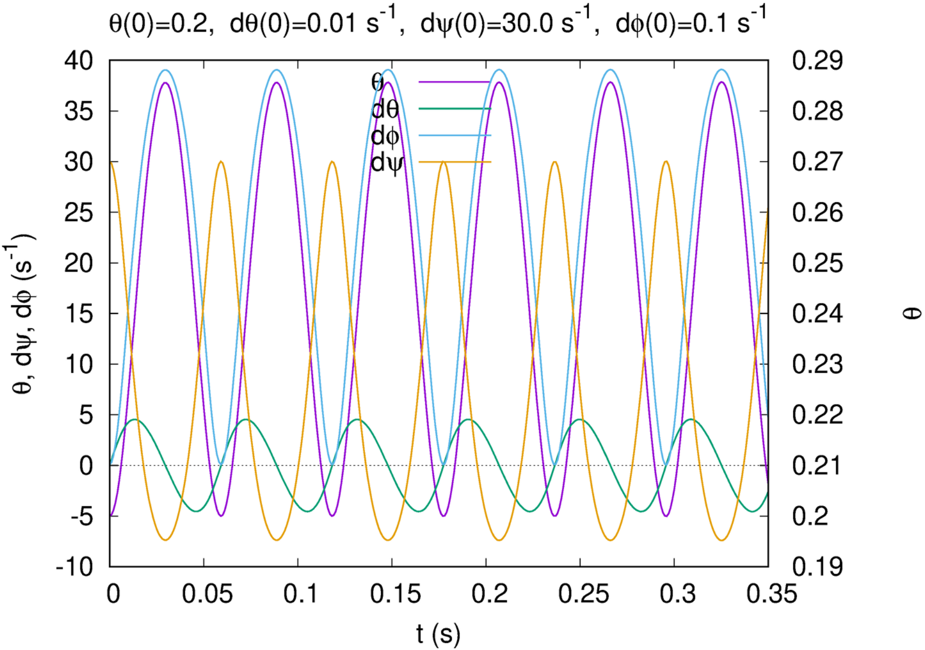
Nokkur gröf birtast á næstu síðum

Umfjöllunin í Goldstein et. al, Third Ed. Classical Mechanics, hjálpar við val á upphafsgildum

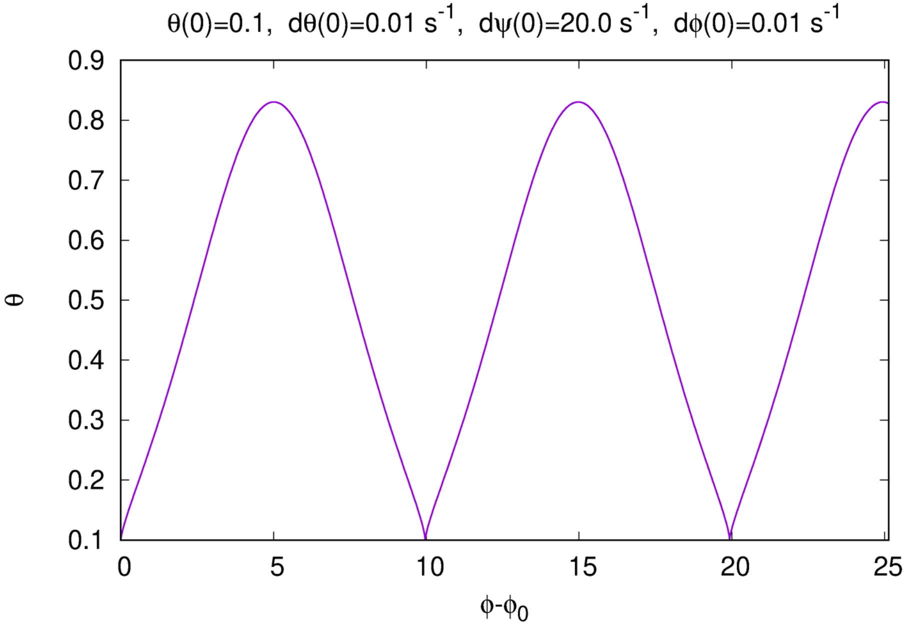

θ(0)=0.2, dθ(0)=0.01 s-1, dψ(0)=10.0 s-1, dϕ(0)=0.1 s-1
θ
dθ
dϕ
dψ
θ, dψ, dϕ (s-1)
θ
t (s)
120
100
80
60
40
20
0
-20
-40
-60
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35

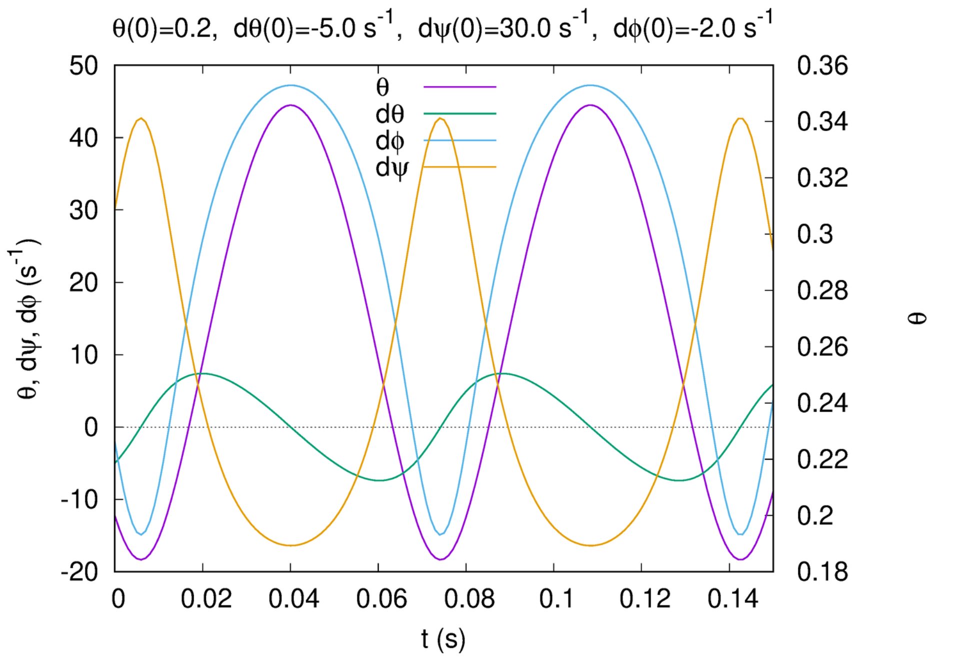


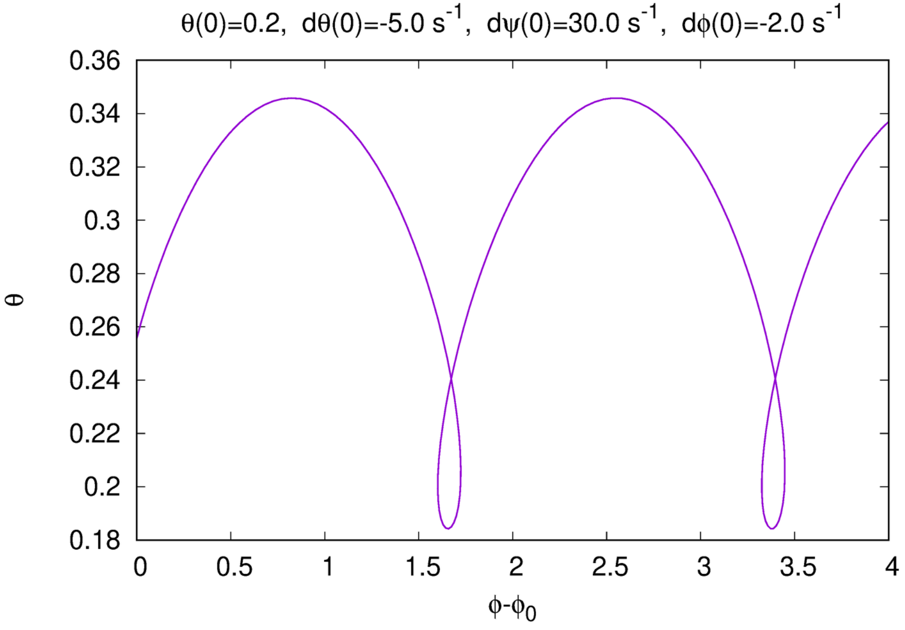
θ(0)=0.2, dθ(0)=0.01 s-1, dψ(0)=20.0 s-1, dϕ(0)=0.1 s-1
θ
dθ
dϕ
dψ
θ, dψ, dϕ (s-1)
θ
t (s)
60
50
40
30
20
10
0
-10
-20
-30
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35

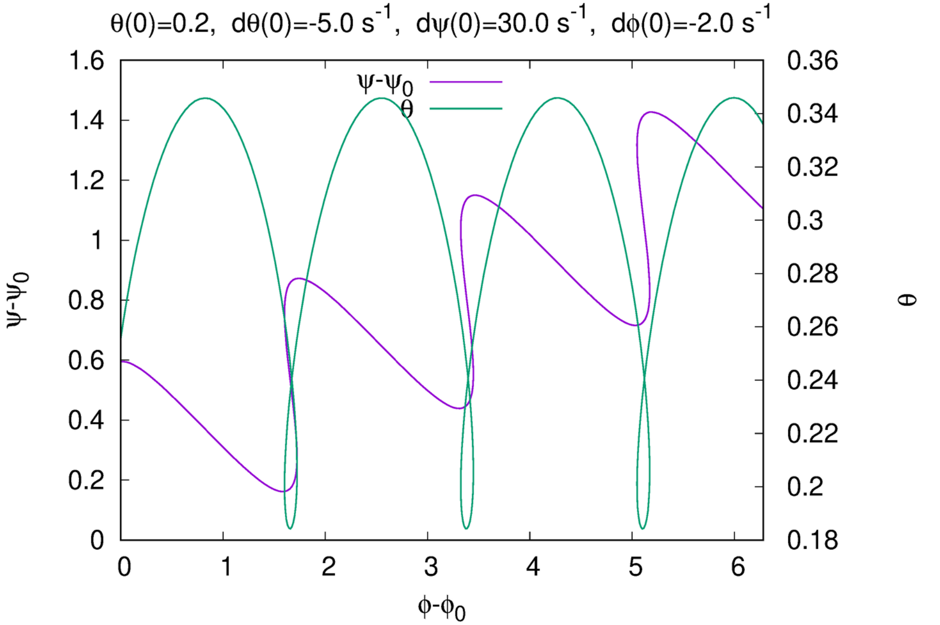


θ(0)=0.2, dθ(0)=0.01 s^-1, dψ(0)=30.0 s^-1, dϕ(0)=0.1 s^-1
θ
dθ
dϕ
dψ
θ, dψ, dϕ (s^-1)
θ
t (s)
40
35
30
25
20
15
10
5
0
-5
-10
0.29
0.28
0.27
0.26
0.25
0.24
0.23
0.22
0.21
0.2
0.19
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35

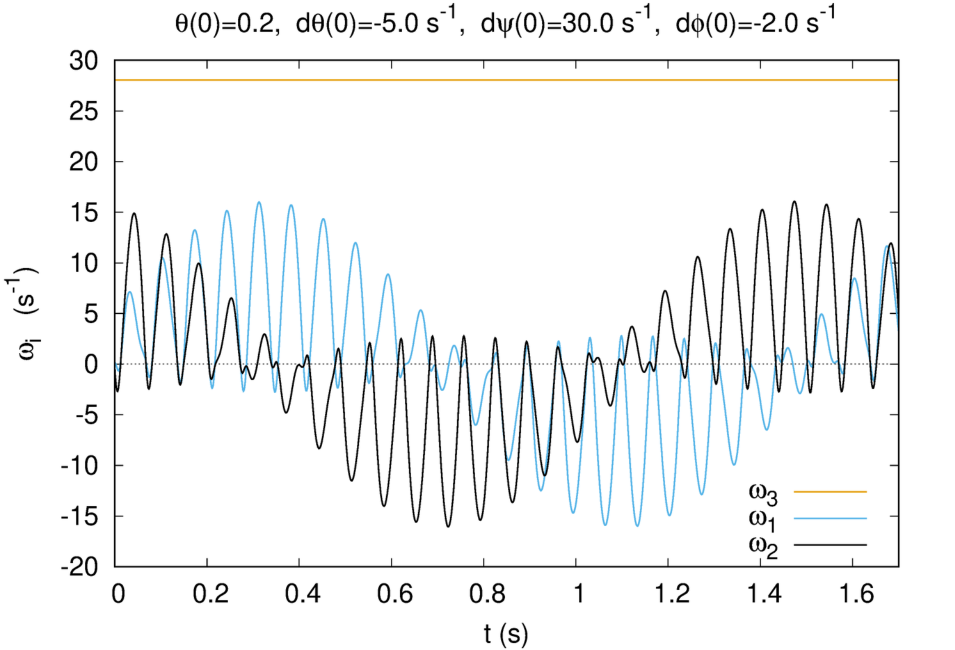

θ(0)=0.1, dθ(0)=0.01 s-1, dψ(0)=20.0 s-1, dϕ(0)=0.01 s-1
θ
ϕ-ϕ0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
5
10
15
20
25


θ(0)=0.2, dθ(0)=-5.0 s-1, dψ(0)=30.0 s-1, dϕ(0)=-2.0 s-1
θ
dθ
dϕ
dψ
θ, dψ, dϕ (s-1)
θ
t (s)
50
40
30
20
10
0
-10
-20
0.36
0.34
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14

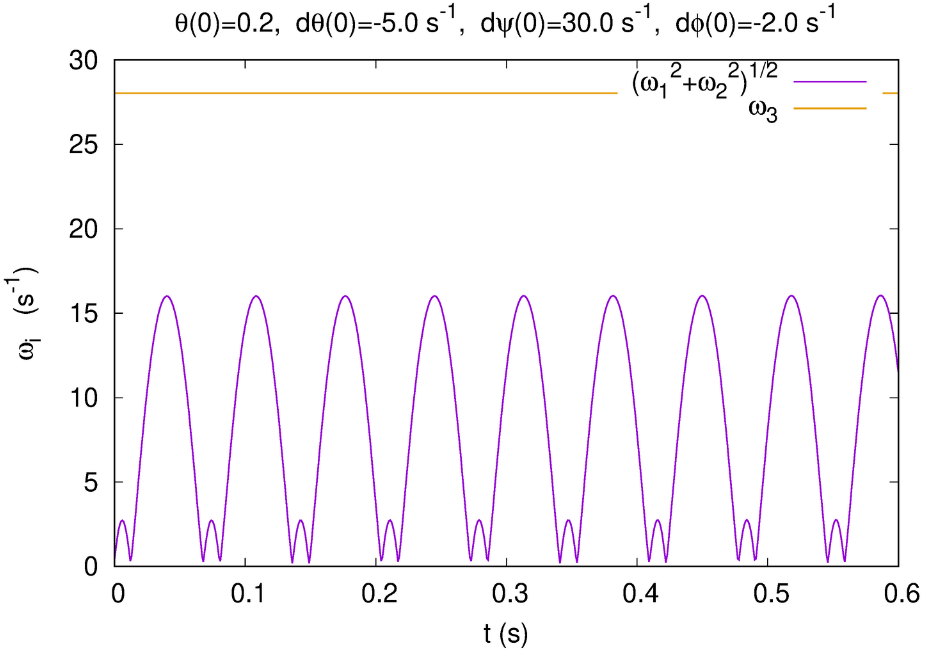

θ(0)=0.2, dθ(0)=-5.0 s-1, dψ(0)=30.0 s-1, dφ(0)=-2.0 s-1
0.36
0.34
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
θ
0
0.5
1
1.5
2
2.5
3
3.5
4
φ-φ0



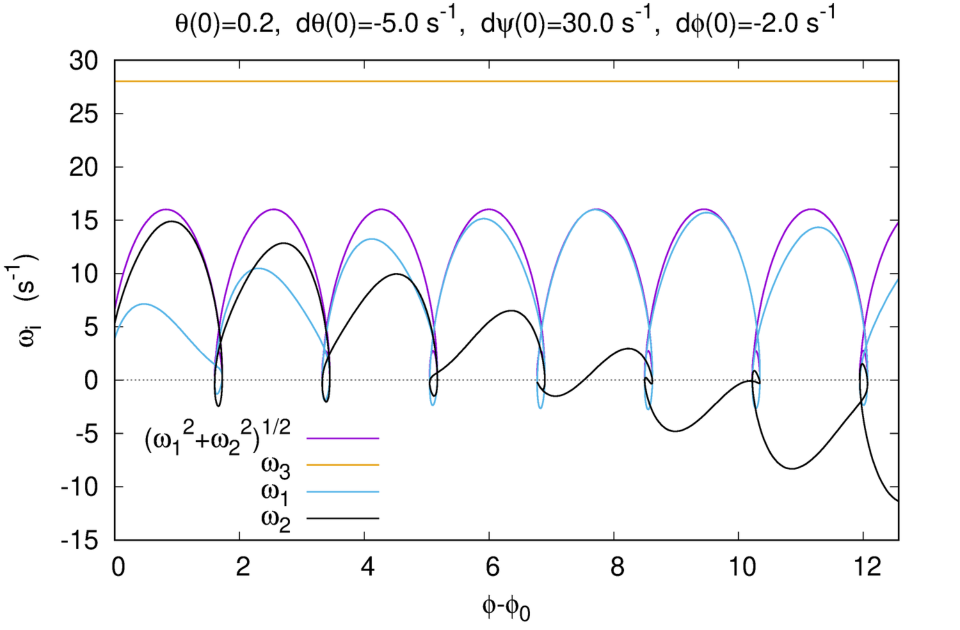
θ(0)=0.2, dθ(0)=-5.0 s-1, dψ(0)=30.0 s-1, dϕ(0)=-2.0 s-1
ψ-ψ0
θ
ϕ-ϕ0
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.36
0.34
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0
1
2
3
4
5
6



θ(0)=0.2, dθ(0)=-5.0 s-1, dψ(0)=30.0 s-1, dϕ(0)=-2.0 s-1
ωi (s-1)
t (s)
ω3
ω1
ω2
30
25
20
15
10
5
0
-5
-10
-15
-20
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6



θ(0)=0.2, dθ(0)=-5.0 s-1, dψ(0)=30.0 s-1, dφ(0)=-2.0 s-1
(ω1^2+ω2^2)^1/2
ω3
ωi (s-1)
t (s)
0
5
10
15
20
25
30
0
0.1
0.2
0.3
0.4
0.5
0.6



θ(0)=0.2, dθ(0)=-5.0 s-1, dψ(0)=30.0 s-1, dϕ(0)=-2.0 s-1
ωi (s-1)
ϕ-ϕ0
(ω1^2+ω2^2)^1/2
ω3
ω1
ω2
30
25
20
15
10
5
0
-5
-10
-15
0
2
4
6
8
10
12