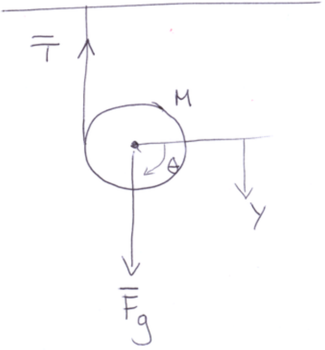
Aflfræði stjarfhluta

Lærum nýja aðferð í aflfræði hér, en til upprifjunar skoðum við fyrst tvö dæmi

1 Fallandi trissa



CM-hreyfing

$$\boxed{\begin{aligned} M\ddot{y} &= F_g - T \\ &= Mg - T \end{aligned}}$$

Hringsnúningurinn er vegna $\bar{T}$

$$\boxed{\tau = RT = I\ddot{\theta}}, \quad I = \frac{1}{2}MR^2$$

$$T = \frac{I\ddot{\theta}}{R} = \frac{1}{2}MR\ddot{\theta}$$

Upphafsskilyrði

$$y(0) = 0$$

$$\theta(0) = 0$$

$$y = R\theta \;\rightarrow\; \dot{y} = V = R\dot{\theta}$$

$$\ddot{y} = g - \frac{T}{M} = g - \frac{I\ddot{\theta}}{MR}$$

$$= g - \frac{1}{2}R\ddot{\theta} = g - \frac{\ddot{y}}{2}$$

$$\rightarrow \ddot{y} = g - \frac{\ddot{y}}{2} \quad \text{eða} \quad \underline{\ddot{y} = \frac{2}{3} g}$$

$$\underline{\alpha = \ddot{\theta}} = \frac{\ddot{y}}{R} = \underline{\frac{2g}{3R}}$$

T má síðan finna

$$\underline{T = \frac{I}{R}\ddot{\theta} = \frac{Mg}{3}}$$

$$\hookrightarrow \quad V = \dot{y} = \frac{2gt}{3}$$

$$\omega = \frac{2gt}{3R}$$

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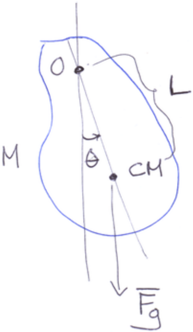
(2) Raunpendúll



$$I = Mk^2$$

$\uparrow$ tregðuarmur

$$T = \frac{1}{2} I \dot{\theta}^2$$

smáar sveiflur

$$U = -MgL\cos\theta \simeq \overbrace{-MgL\left\{1 - \frac{\theta^2}{2}\right\}}$$

$$L = T - U = \frac{I}{2}\dot{\theta}^2 + MgL\left\{1 - \frac{\theta^2}{2}\right\}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\rightarrow \boxed{\ddot{\theta} + \frac{MgL}{I}\theta = 0}$$

$$\omega^2 = \frac{MgL}{I} = \frac{gL}{k^2}$$

Tregðuþinur (Inertiatensor)

Byrjum með stjarfhlut settan saman úr n ögnum með massa m_α, $\alpha = 1, 2, \dots n$

Notum fast hnitakerfi óháð hlut (fixed), og annað fast í hlutnum (r)

$$v_{f,\alpha} = \bar{V} + \bar{v}_{r,\alpha} + \bar{\omega} \times \bar{r}_\alpha$$

agnirnar eru fastar í hnitakerfi hlutar

$$\rightarrow \bar{v}_{r,\alpha} = \left(\frac{d\bar{r}_\alpha}{dt}\right)_{rot} \equiv 0$$

$$\rightarrow \boxed{\bar{v}_\alpha = \bar{V} + \bar{\omega} \times \bar{r}_\alpha}$$

hraði í fasta kerfinu (fixed

fyrir hverja ögn

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2$$

Í heild fyrir stjarfhlutinn

$$T = \frac{1}{2} \sum_\alpha m_\alpha \left\{ \bar{V} + \bar{\omega} \times \bar{r}_\alpha \right\}^2$$

$$= \frac{1}{2} \sum_\alpha m_\alpha \left\{ V^2 + (\bar{\omega} \times \bar{r}_\alpha)^2 + 2\bar{V} \cdot (\bar{\omega} \times \bar{r}_\alpha) \right\}$$

en

$$\sum_\alpha m_\alpha \bar{r}_\alpha = M\bar{R} \; \boxed{= 0}$$

í hnitakerfi stjarfhlutar með miðju í CM

$$\sum_\alpha m_\alpha \bar{V}\cdot(\bar{\omega}\times\bar{r}_\alpha) = \bar{V}\cdot\Big(\bar{\omega}\times\underbrace{\sum_\alpha m_\alpha \bar{r}_\alpha}_{=0}\Big)$$

$$\sum_\alpha m_\alpha V^2 = V^2\sum_\alpha m_\alpha = MV^2$$

$$\rightarrow \boxed{T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_\alpha m_\alpha(\bar{\omega}\times\bar{r}_\alpha)^2 = T_{trans} + T_{rot}}$$

Although

$$(\bar{A}\times\bar{B})^2 = (\bar{A}\times\bar{B})\cdot(\bar{A}\times\bar{B}) = \left\{A^2B^2\sin^2\theta = A^2B^2(1-\cos^2\theta)\right\}$$

$$= A^2B^2 - (\bar{A}\cdot\bar{B})^2$$

$$\rightarrow T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha\left\{\omega^2 r_\alpha^2 - (\bar{\omega}\cdot\bar{r}_\alpha)^2\right\}$$

umritum

$$T_{rot} = \frac{1}{2}\sum_{\alpha} m_\alpha \left[\omega^2 r_\alpha^2 - (\bar{\omega}\cdot\bar{r}_\alpha)^2\right] = \frac{1}{2}\sum_{\alpha} m_\alpha \left\{\left(\sum_i \omega_i^2\right)\left(\sum_k x_{\alpha,k}^2\right) - \left(\sum_i \omega_i x_{\alpha,i}\right)\left(\sum_j \omega_j x_{\alpha,j}\right)\right\}$$

notum

$$\omega_i = \sum_j \omega_j \delta_{i,j}$$

$$\rightarrow T_{rot} = \frac{1}{2}\sum_{\alpha}\sum_{i,j} m_\alpha \left\{\omega_i \omega_j \delta_{ij}\left(\sum_k x_{\alpha,k}^2\right) - \omega_i \omega_j x_{\alpha,i} x_{\alpha,j}\right\}$$

$$= \frac{1}{2}\sum_{i,j} \omega_i \omega_j \sum_{\alpha} m_\alpha \left\{\delta_{ij}\sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right\}$$

Ef

$$\boxed{I_{ij} \equiv \sum_{\alpha} m_\alpha \left\{\delta_{i,j}\sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right\}} \rightarrow \boxed{T_{rot} = \frac{1}{2}\sum_{i,j} I_{ij}\,\omega_i \omega_j}$$

$\mathbb{I}$ með stök I_{ij} lítur út fyrir að vera fylki (3×3)
Við munum síðar komast að því að $\mathbb{I}$ er þinur (tensor)
{vegna þess hvernig hann ummyndast milli hnitakerfa}

$\mathbb{I}$ er tregðuþinur (inertia tensor) (í bókinni er $\mathbb{I} = \{I\}$)

$$I_{ij} = \sum_\alpha m_\alpha \left\{ \delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i}\, x_{\alpha,j} \right]$$

$$\mathbb{I} = \begin{Bmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha x_{\alpha,1} x_{\alpha,2} & -\sum_\alpha m_\alpha x_{\alpha,1} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,3} x_{\alpha,1} & -\sum_\alpha m_\alpha x_{\alpha,3} x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{Bmatrix}$$

Þetta má umrita með $(x_\alpha, y_\alpha, z_\alpha) = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$

$$r_\alpha^2 = x_\alpha^2 + y_\alpha^2 + z_\alpha^2$$

$$\mathbb{I} = \begin{Bmatrix} \sum_\alpha m_\alpha (r_\alpha^2 - x_\alpha^2) & -\sum_\alpha m_\alpha x_\alpha y_\alpha & -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ -\sum_\alpha m_\alpha y_\alpha x_\alpha & \sum_\alpha m_\alpha (r_\alpha^2 - y_\alpha^2) & -\sum_\alpha m_\alpha y_\alpha z_\alpha \\ -\sum_\alpha m_\alpha z_\alpha x_\alpha & -\sum_\alpha m_\alpha z_\alpha y_\alpha & \sum_\alpha m_\alpha (r_\alpha^2 - z_\alpha^2) \end{Bmatrix}$$

I_{11}, I_{22} og I_{33} eru tregðuvægin um x-, y-, eða z-ás
(Moments of inertia)

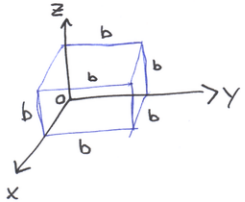
I_{ij} með $i \neq j$ eru tregðumargfeldin (products of inertia) þinurinn er samhverfur með $I_{ij} = I_{ji}$

$\rightarrow$ $\mathbb{I}$ er með 6 óháð stök, hann er samleggjanlegur fyrir massapunktana

því fæst fyrir massadreifingu $\rho(\vec{r})$

$$\boxed{I_{ij} = \int_V dv\, \rho(\vec{r}) \left[\delta_{ij} \sum_k x_k^2 - x_i x_j \right]}$$

Sýnidæmi



Teningur með fasta dreifingu ρ, $M = \rho b^3$
notum hnitakerfið á myndinni, hér er
O ekki í CM

$$I_{11} = \rho \int_0^b dz \int_0^b dy\,(y^2+z^2) \int_0^b dx = \frac{2}{3}\rho b^5 = \frac{2}{3} M b^2$$

$$I_{12} = -\rho \int_0^b dx\, x \int_0^b dy\, y \int_0^b dz = -\frac{1}{4}\rho b^5 = -\frac{1}{4} M b^2$$

Skilgreinum $\beta \equiv Mb^2$ þá fæst

$$I_{11} = I_{22} = I_{33} = \frac{2}{3}\beta$$

$$I_{12} = I_{13} = I_{23} = -\frac{1}{4}\beta$$

og $$\mathbb{I} = \begin{pmatrix} \frac{2}{3}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta \end{pmatrix}$$

Hverfiþungi

$$L = \sum_\alpha \bar{r}_\alpha \times \bar{p}_\alpha$$

miðað við O í hnitakerfi hlutar
venjulega er O valinn sem punktur
sem er kyrr í ytra fastakerfinu
eða CM hlutar

$$\bar{p}_\alpha = m_\alpha \bar{v}_\alpha = m_\alpha (\bar{\omega} \times \bar{r}_\alpha)$$

$$\rightarrow L = \sum_\alpha m_\alpha \left\{ \bar{r}_\alpha \times (\bar{\omega} \times \bar{r}_\alpha) \right\}$$

notum $\bar{A} \times (\bar{B} \times \bar{A}) = A^2 \bar{B} - \bar{A}(\bar{A} \cdot \bar{B})$

$$\rightarrow \boxed{L = \sum_\alpha m_\alpha \left\{ r_\alpha^2 \bar{\omega} - \bar{r}_\alpha (\bar{r}_\alpha \cdot \bar{\omega}) \right\}}$$

með sömu aðferð og áður fæst

$$L_i = \sum_\alpha m_\alpha \left\{ \omega_i \sum_k x_{\alpha,k}^2 - x_{\alpha,i} \left(\sum_j x_{\alpha,j} \omega_j \right) \right\}$$

$$\boxed{L_i = \sum_\alpha m_\alpha \sum_j \left\{ \omega_j \delta_{ij} \sum_k x_{\alpha,k}^2 - \omega_j x_{\alpha,i} x_{\alpha,j} \right\}}$$

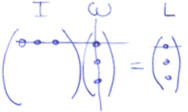
$$= \sum_j \omega_j \sum_\alpha m_\alpha \left\{ \delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right\} \quad \boxed{= \sum_j I_{ij} \omega_j}$$

því fæst

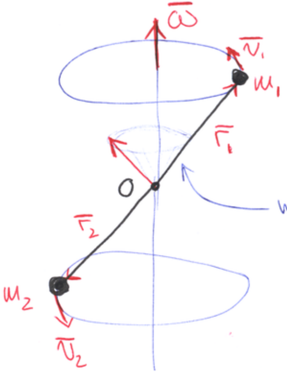
$$\boxed{\bar{L} = \mathbb{I} \cdot \bar{\omega}}$$

því er opnað fyrir þann möguleika að $\bar{\omega}$ og $\bar{L}$ séu ekki alltaf samsíða

→ lítur út eins og margfeldi fylkis og dálkvigurs



Skoðum dæmi



$$\bar{v}_\alpha = \bar{\omega} \times \bar{r}_\alpha$$

$$\bar{L} = \sum_\alpha m_\alpha (\bar{r}_\alpha \times \bar{v}_\alpha)$$

$\bar{L}$ er hornrétt á stöngina
ekki fasti (stefnan),
snýst um snúningsás
og strikar keiluflöt

Enda þarf vægi til að viðhalda föstum snúningi

$$\dot{\bar{L}} = \bar{N}$$

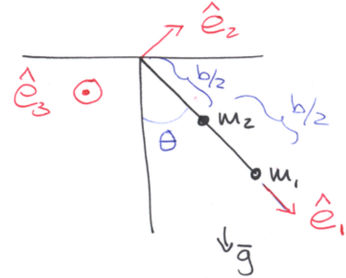
Einnig

$$\frac{1}{2}\sum_i \omega_i L_i = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j = T_{rot}$$

$$\rightarrow T_{rot} = \frac{1}{2}\bar{\omega}\cdot\bar{L}$$

$$= \frac{1}{2}\bar{\omega}\cdot\mathbb{I}\cdot\bar{\omega}$$

Annað dæmi



Massalaus stöng með tvo massa

$$\bar{\omega} = \omega_3\hat{e}_3 = \dot{\theta}\hat{e}_3$$

Allur massinn liggur í $\hat{e}_1$-stefnu

$$x_{2,1} = \frac{b}{2} \quad \text{og} \quad x_{1,1} = b$$

Öll önnur hnit $x_{\alpha,k}$ hverfa

$$I_{ij} = m_1\left\{\delta_{ij}x_{1,1}^2 - x_{1,i}x_{1,j}\right\} + m_2\left\{\delta_{ij}x_{2,1}^2 - x_{2,i}x_{2,j}\right\}$$

$$\mathbb{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_1 b^2 + m_2 \frac{b^2}{4} & 0 \\ 0 & 0 & m_1 b^2 + m_2 \frac{b^2}{4} \end{bmatrix}$$

$$L_i = \sum_j I_{ij} \omega_j \longrightarrow \begin{cases} L_1 = 0 \\ L_2 = 0 \\ L_3 = I_{33}\omega_3 = \left\{ m_1 b^2 + m_2 \frac{b^2}{4} \right\} \dot{\theta} \end{cases}$$

$$\dot{\bar{L}} = \bar{N}$$

$$\longrightarrow \left\{ m_1 b^2 + m_2 \frac{b^2}{4} \right\} \ddot{\theta}\, \hat{e}_3 = \sum_\alpha \bar{r}_\alpha \times \bar{F}_\alpha$$

$$\bar{g} = g \cos\theta \cdot \hat{e}_1 - g \sin\theta \cdot \hat{e}_2$$

$$\bar{r}_1 \times \bar{F}_1 = b\hat{e}_1 \times (\cos\theta\, \hat{e}_1 - \sin\theta\, \hat{e}_2) m_1 g = -m_1 g b \sin\theta \cdot \hat{e}_3$$

$$\bar{r}_2 \times \bar{F}_2 = \frac{b}{2} \hat{e}_1 \times (\cos\theta\, \hat{e}_1 - \sin\theta\, \hat{e}_2) m_2 g = -m_2 g \frac{b}{2} \sin\theta \cdot \hat{e}_3$$

því verður hreyfijafnan

$$b^2\left\{m_1+\frac{m_2}{4}\right\}\ddot{\theta} = -bg\sin\theta\cdot\left\{m_1+\frac{m_2}{2}\right\}$$

$$\rightarrow \omega_0^2 = \frac{\left(m_1+\frac{m_2}{2}\right)}{\left(m_1+\frac{m_2}{4}\right)}\frac{g}{b}$$

Auðvitað hefði verið styttra hér að nota
T og U $\rightarrow$ L og Euler-Lagrange

Ef $m_1 \gg m_2 \rightarrow \omega_0^2 = \frac{m_1}{m_1}\frac{\left(1+\frac{m_2}{2m_1}\right)}{\left(1+\frac{m_2}{4m_1}\right)}\frac{g}{b}$

$$\approx \frac{g}{b}$$ eins og búast mátti við