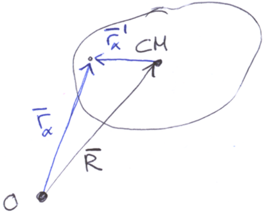
Hverfiþungi í agnakerfi



Heppilegt er að greina staðarvigur agnar α í

massamiðju staðsetn.

$$\bar{r}_\alpha = \bar{R} + \bar{r}'_\alpha$$

m.v. CM

Hverfiþungi agnar α

$$\bar{L}_\alpha = \bar{r}_\alpha \times \bar{p}_\alpha$$

-> Heildar hverfiþungi

$$\bar{L} = \sum_\alpha \bar{L}_\alpha = \sum_\alpha \left\{ \bar{r}_\alpha \times m_\alpha \dot{\bar{r}}_\alpha \right\}$$

notum

$$\bar{L} = \sum_\alpha (\bar{r}'_\alpha + \bar{R}) \times m_\alpha (\dot{\bar{r}}'_\alpha + \dot{\bar{R}})$$

$$= \sum_\alpha m_\alpha \left\{ (\bar{r}'_\alpha \times \dot{\bar{r}}'_\alpha) + \overbrace{(\bar{r}'_\alpha \times \dot{\bar{R}}) + (\bar{R} \times \dot{\bar{r}}'_\alpha)}^{\text{skoðum}} + (\bar{R} \times \dot{\bar{R}}) \right\}$$

1

$$1 = \left\{\sum_\alpha m_\alpha \bar{r}_\alpha'\right\} \times \dot{\bar{R}} + \bar{R} \times \frac{d}{dt}\left\{\sum_\alpha m_\alpha \bar{r}_\alpha'\right\}$$

en

$$\sum_\alpha m_\alpha \bar{r}_\alpha' = \sum_\alpha m_\alpha(\bar{r}_\alpha - \bar{R}) = \sum_\alpha m_\alpha \bar{r}_\alpha - \bar{R}\sum_\alpha m_\alpha$$

$$= M\bar{R} - \bar{R}M = 0$$

massamiðja kerfis í massamiðjuhnitum

$$\rightarrow \bar{L} = M\bar{R} \times \dot{\bar{R}} + \sum_\alpha \bar{r}_\alpha' \times \bar{p}_\alpha' = \underbrace{\bar{R} \times \bar{P}}_{} + \underbrace{\sum_\alpha \bar{r}_\alpha' \times \bar{p}_\alpha'}_{}$$

→ Heildarhverfiþunginn

= Hverfiþungi CM um O

+ Summa hverfiþunga hverrar agnar um CM

Skoðum breytingar á hverfiþunga

$$m_\alpha \dot{\bar{r}} \times \dot{\bar{r}} = 0$$

$$\dot{L}_\alpha = \dot{\bar{r}}_\alpha \times \bar{p}_\alpha + \bar{r}_\alpha \times \dot{\bar{p}}$$

$$= \bar{r}_\alpha \times \left\{ \bar{F}_\alpha^{(e)} + \sum_\beta \bar{f}_{\alpha\beta} \right\}$$

fyrir heildina

$\rightarrow$

$$\dot{L} = \sum_\alpha \dot{L}_\alpha = \sum_\alpha \left\{ \bar{r}_\alpha \times \bar{F}_\alpha^{(e)} \right\} + \sum_{\alpha,\beta\neq\alpha} \left\{ \bar{r}_\alpha \times \bar{f}_{\alpha\beta} \right\}$$

$$= \sum_{\alpha<\beta} \left\{ (\bar{r}_\alpha \times \bar{f}_{\alpha\beta}) + (\bar{r}_\beta \times \bar{f}_{\beta\alpha}) \right\}$$

notum $\bar{r}_{\alpha\beta} \equiv \bar{r}_\alpha - \bar{r}_\beta$ og 3. lögmálið $\bar{f}_{\alpha\beta} = -\bar{f}_{\beta\alpha}$

$$\sum_{\alpha\beta\neq\alpha} \left\{ \bar{r}_\alpha \times \bar{f}_{\alpha\beta} \right\} = \sum_{\alpha<\beta} (\bar{r}_\alpha - \bar{r}_\beta) \times \bar{f}_{\alpha\beta} = \sum_{\alpha<\beta} (\bar{r}_{\alpha\beta} \times \bar{f}_{\alpha\beta})$$

En ef við notum sterku útgáfu 3. lögmálsins

þ.a. $\bar{F}_{\alpha\beta}$ sé í stefnu $\pm\bar{r}_{\alpha\beta}$ fæst $\bar{r}_{\alpha\beta}\times\bar{f}_{\alpha\beta}=0$

$$\rightarrow \quad \dot{\bar{L}} = \sum_\alpha \left\{\bar{r}_\alpha \times \bar{F}_\alpha^{(e)}\right\} = \sum_\alpha \bar{N}_\alpha^{(e)} = \bar{N}^{(e)}$$

$\rightarrow$ Ef heildar ytra vægið á agnakerfi um ás er 0 $\rightarrow$ hverfiþungi kerfisins um ásinn er fastur

Einnig

$$\sum_{\alpha,\beta\neq\alpha}(\bar{r}_\alpha\times\bar{f}_{\alpha\beta}) = \sum_{\alpha<\beta}(\bar{r}_{\alpha\beta}\times\bar{F}_{\alpha\beta}) = 0$$

$\rightarrow$ heildar innra vægið $=0$ ef innri kraftarnir eru miðlægir. $\bar{L}$-i verður ekki breytt án ytri krafta

Orka agnakerfis

Hugsum tvö ástönd agnakerfis, 1 og 2.
Til þess að breyta ástandi kerfisins þarf vinnu
t.a. flytja agnir til, þá er

$$W_{12} = \sum_{\alpha} \int_1^2 \bar{F}_\alpha \cdot d\bar{r}_\alpha$$

þar sem $\bar{F}_\alpha$ er heildarkrafturinn á ögn α

Skoðum fyrirfram hvað við sjáum

Við getum heildað og fundið W_{12} á mismunandi hátt
Ef ytri og innri kraftar eru geymnir þá sjáum við
að lotum að heildarorkan er varðveitt, $\Delta T = -\Delta U$

En, kerfi t.d. tvær eindir tengdar með gormi

Ef við teygjum á gorminum þá getur $\Delta T = 0$, en $\Delta U \neq 0$ ..., en okkar kraftar er ekki geyminn og .... lokað, opið .... (massarnir og gormurinn í þyngdarmætti er geymdir)

Byrjum því aftur

$$\bar{F} \cdot d\bar{r} = m \frac{d\bar{v}}{dt} \cdot \frac{d\bar{r}}{dt} dt = m \frac{d\bar{v}}{dt} \cdot \bar{v} dt = \frac{m}{2} \frac{d}{dt} (\bar{v} \cdot \bar{v}) dt = \frac{m}{2} \frac{d}{dt} (v^2) dt = d\left\{\tfrac{1}{2} m v^2\right\} \ldots$$

$$W_{12} = \sum_\alpha \int_1^2 \bar{F}_\alpha \cdot d\bar{r}_\alpha$$

$$= \sum_\alpha \int_1^2 d\left\{\tfrac{1}{2} m_\alpha v_\alpha^2\right\} = T_2 - T_1$$

notum $\dot{\bar{r}}_\alpha = \dot{\bar{r}}'_\alpha + \dot{\bar{R}}$ $\rightarrow$ $\dot{\bar{r}}_\alpha \cdot \dot{\bar{r}}_\alpha = v_\alpha^2 = (\dot{\bar{r}}'_\alpha + \dot{\bar{R}}) \cdot (\dot{\bar{r}}'_\alpha + \dot{\bar{R}})$

$$= (\dot{\bar{r}}'_\alpha \cdot \dot{\bar{r}}'_\alpha) + 2(\dot{\bar{r}}_\alpha \cdot \dot{\bar{R}}) + (\dot{\bar{R}} \cdot \dot{\bar{R}})$$

$$= v'^2_\alpha + 2(\dot{\bar{r}}'_\alpha \cdot \dot{\bar{R}}) + V^2$$

$\bar{v}' = \dot{\bar{r}}'$

$\bar{V}$ er CM hraði

$$T = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 = \sum_\alpha \frac{m_\alpha}{2} v_\alpha'^2 + \sum \frac{m_\alpha}{2} V^2 + \dot{\bar{R}} \cdot \frac{d}{dt} \underbrace{\sum_\alpha m_\alpha \bar{r}_\alpha'}_{=0}$$

$$\rightarrow \boxed{T = \sum \frac{m_\alpha}{2} v_\alpha'^2 + \frac{1}{2} M V^2}$$

Hreyfiorka kerfis er jöfn summu hreyfiorku CM og hreyfiorku hverrar agnar miðað við CM

skoðum aftur

heildar

$$W_{12} = \sum_\alpha \int_1^2 \bar{F}_\alpha \cdot d\bar{r}_\alpha$$

ytri   innri

$$= \sum_\alpha \int_1^2 \bar{F}_\alpha^{(e)} \cdot d\bar{r}_\alpha + \sum_{\alpha,\beta\neq\alpha} \int_1^2 \bar{f}_{\alpha\beta} \cdot d\bar{r}_\alpha$$

Kraftarnir eru geymnir

m.t.t. $x_{\alpha,i}$

$$\bar{F}_\alpha^{(e)} = -\bar{\nabla}_\alpha U_\alpha$$

$$\bar{f}_{\alpha\beta} = -\bar{\nabla}_\alpha \bar{U}_{\alpha\beta}$$

ekki sömu skalar föllin

$$\sum_\alpha \int_1^2 \bar{F}^{(e)} \cdot d\bar{r}_\alpha = -\sum_\alpha \int_1^2 (\bar{\nabla}_\alpha U_\alpha) \cdot d\bar{r}_\alpha = -\sum_\alpha U_\alpha \Big|_1^2$$

$$\sum_{\alpha,\beta \neq \alpha} \int_1^2 \bar{f}_{\alpha\beta} \cdot d\bar{r}_\alpha = \sum_{\alpha<\beta} \int_1^2 (\bar{f}_{\alpha\beta} \cdot d\bar{r}_\alpha + \bar{f}_{\beta\alpha} \cdot d\bar{r}_\beta)$$

$$= \sum_{\alpha<\beta} \int_1^2 \bar{f}_{\alpha\beta} \cdot \underbrace{(d\bar{r}_\alpha - d\bar{r}_\beta)} = \sum_{\alpha<\beta} \int_1^2 \bar{f}_{\alpha\beta} \cdot \underbrace{d\bar{r}_{\alpha\beta}}$$

$$\bar U_{\alpha\beta} = \bar U_{\alpha\beta}(|\bar r_\alpha - \bar r_\beta|)$$

but $\bar U_{\alpha\beta} = \bar U_{\beta\alpha}$

$$\to d\bar U_{\alpha\beta} = \sum_i \left\{ \frac{\partial \bar U_{\alpha\beta}}{\partial x_{\alpha,i}}\, dx_{\alpha,i} + \frac{\partial \bar U_{\alpha\beta}}{\partial x_{\beta,i}}\, dx_{\beta,i} \right\}$$

$$= \underbrace{\left(\bar\nabla_\alpha \bar U_{\alpha\beta}\right)}_{-\bar f_{\alpha\beta}} \cdot d\bar r_\alpha + \underbrace{\left(\bar\nabla_\beta \bar U_{\alpha\beta}\right)}_{\substack{= \bar\nabla_\beta \bar U_{\beta\alpha} \\ = -\bar f_{\beta\alpha} = \bar f_{\alpha\beta}}} \cdot d\bar r_\beta$$

$$\to = -\bar f_{\alpha\beta} \cdot (d\bar r_\alpha - d\bar r_\beta)$$

$$= -\bar f_{\alpha\beta} \cdot d\bar r_{\alpha\beta}$$

$$\to \sum_{\alpha,\beta\neq\alpha} \int_1^2 \bar f_{\alpha\beta} \cdot d\bar r_\alpha = -\sum_{\alpha<\beta} \int_1^2 d\bar U_{\alpha\beta} = -\sum_{\alpha<\beta} \bar U_{\alpha\beta} \Big|_1^2$$

því fæst

$$W_{12} = -\sum_{\alpha} U_\alpha \Big|_1^2 - \sum_{\alpha<\beta} \bar{U}_{\alpha\beta} \Big|_1^2$$

því er til <u>heildarmættisorka</u> kerfisins

$$U = \sum_{\alpha} U_\alpha + \overbrace{\sum_{\alpha<\beta} \bar{U}_{\alpha\beta}}^{\text{innri orka}}$$

$$\rightarrow W_{12} = -U\Big|_1^2 = U_1 - U_2$$

og með fyrri niðurstöðu

$$\boxed{T_2 - T_1 = U_1 - U_2}$$

eða

$$T_1 + U_1 = T_2 + U_2$$

$$\boxed{E_1 = E_2}$$

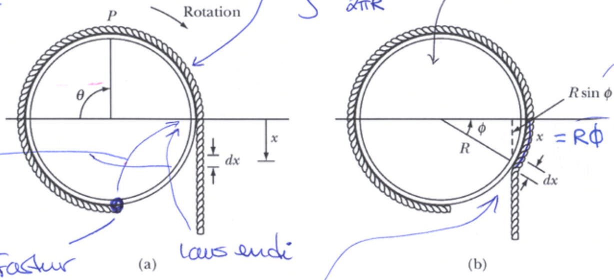
Heildarorka geymins kerfis er föst

Dæmi, reipi á hjóli

í þyngdarsviði

Reipi $\rho = \frac{m}{2\pi R}$

hjól: M



upphafsstaða

fastur endi

laus endi

$x = R\phi$ bogalengd

Finnum hamferð hjóls

þyngdarkraftar framkvæma vinnu

Hversu mikla vinnu þurfum við að framkvæma til að vefja reipið aftur að hjólinu?

Hve langt þarf að lyfta dx?

$$= \text{bogalengd} - R\sin\phi$$

$$= x - R\sin\left(\frac{x}{R}\right)$$

$$\frac{\text{Vinna}}{\text{þyngdarkrafts fyrir } dx} = \rho\, dx\, g\left\{x - R\sin\left(\frac{x}{R}\right)\right\}$$

Vinna fyrir θ →

$$W(\theta) = \int_0^{R\theta} \rho g\left\{x - R\sin\left(\frac{x}{R}\right)\right\}dx$$

$$W(\theta) = \rho g R^2 \left\{ \frac{\theta^2}{2} + \cos\theta - 1 \right\}$$

þessi vinna þyngdarkrafts fer öll í hreyfiorku

$$T = \underbrace{\frac{1}{2} M (R\dot{\theta})^2}_{\text{hjól}} + \underbrace{\frac{1}{2} m (R\dot{\theta})^2}_{\text{reipi}}$$

$$\rightarrow \frac{mgR}{2\pi} \left\{ \frac{\theta^2}{2} + \cos\theta - 1 \right\} = \frac{1}{2}(m+M)(R\dot{\theta})^2$$

$$\rightarrow \dot{\theta}^2 = \frac{mg(\theta^2 + 2\cos\theta - 2)}{2\pi R(m+M)}$$

hluti á hjólinu, og annar sem hangir. Báðir em hér því v^2 er óháð stefnu og hraði í θ-átt og x-átt er sá sami $R\dot{\theta}$

Ófjaðrandi árekstur

upphaf

$m_1 \longrightarrow \bar{u}_1 \qquad m_2 \longrightarrow \bar{u}_2$

endir

$m_1 \longrightarrow \bar{v}_1 \qquad m_2 \longrightarrow \bar{v}_2$

$$Q + \frac{m_1 u_1^2}{2} + \frac{m_2 u_2^2}{2} = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2}$$

$Q = 0$: fjaðrandi, hreyfiorka er varðveitt (engin mótstaða

$Q > 0$: orkugæfur árekstur, hreyfiorka eykst

$Q < 0$: orkukræfur árekstur, hreyfiorka minnkar

Atlag (impulse)

2. lögmál Newtons gildir allan árekstur inn, en krafturinn er ekki nákvæmlega þekktur

$$\bar{F} = \frac{d}{dt}(m\bar{v})$$

en atlagið

$$\int_{t_1}^{t_2} \bar{F}\, dt \equiv \bar{P} \quad , \quad \Delta t = t_2 - t_1$$

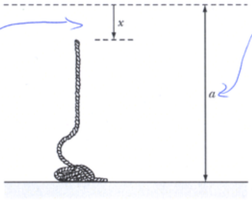
má mæla út frá breytingum á skriðþunga

Skoðum dæmi

Dæmi

Reipi með lengd a og massaþétt ρ fellur á borð

Hver er krafturinn á borðið þegar reipið hefur fallið um x



vegna reipis á borðinu

$$mg = \rho x g$$

en hvað með atlagið eða atlagskraftinn

$$\bar{F}_{impulse} = \frac{dp}{dt}$$

á dt fellur $\rho \overbrace{v dt}^{dx}$ á borðið

$$\to dp = \overbrace{(\rho v dt)}^{dm} v = \rho v^2 dt$$

$$\to \frac{dp}{dt} = \rho v^2 = F_{impulse}$$

fyrir fallið gildir

$$v^2 = 2gx$$

$$\to F_{impulse} = \rho v^2 = 2\rho x g$$

heildarkrafturinn á borðið er því

$$F = F_g + F_{impulse} = \rho x g + 2\rho x g$$

$$\boxed{F = 3\rho x g}$$

sem er 3-faldur kraftur þess sem liggur á borðinu