# Hreyfing reikistjarna – Kepler

$$\theta(r) = \int \frac{\frac{\ell}{r^2}dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right)}} + C$$

$u \equiv \frac{\ell}{r}$, skilgreinum

$\theta$ þ.a. $\theta(r_{min}) = 0$

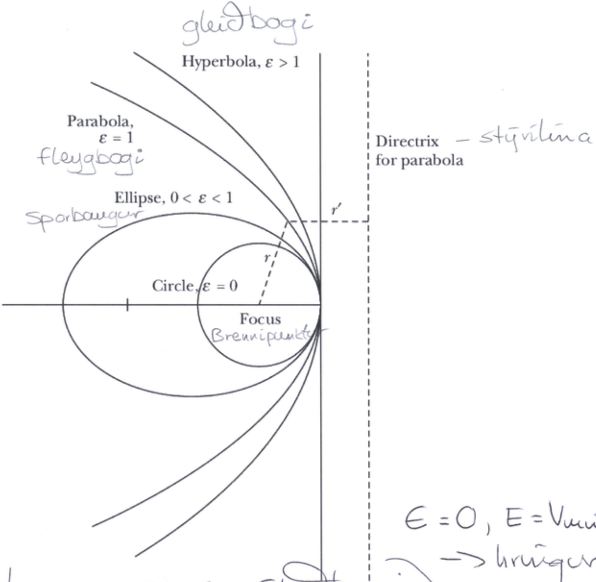
$$\rightarrow \cos\theta = \frac{\frac{\ell^2}{\mu k}\cdot\frac{1}{r} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}}$$

ef $\alpha \equiv \frac{\ell^2}{\mu k}$, $\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$

$$\rightarrow \boxed{\frac{\alpha}{r} = 1 + \epsilon\cos\theta}$$

$\epsilon$: hringvik

$2\alpha$: þverbrennistrengur




$\epsilon = 0$, $E = V_{min}$ → hringur

$\epsilon > 1$, $E > 0$ Gleiðbogi

$\epsilon = 1$, $E = 0$ fleygbogi

$0 < \epsilon < 1$, $V_{min} < E < 0$ sporbaugur

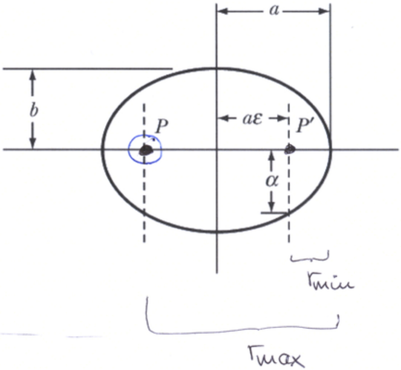
# Reikistjörnur – sporbaugar

Langás $a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2|E|}$

Skammás $b = \frac{\alpha}{\sqrt{1-\epsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}$

Nánd $r_{min} = a(1-\epsilon) = \frac{\alpha}{1+\epsilon}$

Firrð $r_{max} = a(1+\epsilon) = \frac{\alpha}{1-\epsilon}$




## Lota flatarhraði

$$dt = \frac{2\mu}{l} dA$$

Í einni lotu $\tau$ er allur flötur sporbaugsins þekinn

$$\rightarrow \int_0^{\tau} dt = \frac{2\mu}{l} \int_0^{A} dA'$$

$$\rightarrow \tau = \frac{2\mu}{l} A = \frac{2\mu}{l}(\pi a b)$$

$$= \frac{2\mu}{l} \cdot \pi \cdot \frac{k}{2|E|} \cdot \frac{l}{\sqrt{2\mu|E|}}$$

$$= \pi k \sqrt{\frac{\mu}{2}} \, |E|^{-3/2}$$

en $b = \sqrt{\alpha a}$

$$\rightarrow \boxed{\tau^2 = \frac{4\pi^2\mu}{k} a^3}$$

3. lögmál Keplers

Nú er $F(r) = -\frac{G m_1 m_2}{r^2} = -\frac{k}{r^2}$

$$\rightarrow k = G m_1 m_2$$

$$\rightarrow \tau^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)}$$

Lögmál Keplers

1 sporbrautir með sól í öðrum brennipunktinum

2 flatarhraðinn er fasti

3 $\tau^2 = \frac{4\pi^2\mu}{k} a^3$

---

Stöðugleiki hringbrauta

Allar kraftjöfnur
$$F(r) = -\frac{k}{r^n}$$

leifa hringbrautir, en hverjar stöðugar?

Mættið er þá

$$U(r) = -\frac{k}{n-1}\frac{1}{r^{n-1}}$$

Virkamættið er þá

$$V(r) = -\frac{k}{n-1}\cdot\frac{1}{r^{n-1}} + \frac{\ell^2}{2\mu r^2}$$

Skilyrði lágmarks í $V(r)$ og því stöðugri hringbraut er

$$\left.\frac{\partial V}{\partial r}\right|_{r=\rho} = 0 \qquad \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=\rho} > 0$$

þarsem $\rho$ er gildi hringbrautar

$$\left.\frac{\partial V}{\partial r}\right|_{r=\rho} = \frac{k}{\rho^n} - \frac{\ell^2}{\mu\rho^3} = 0$$

$$\rightarrow \boxed{\rho^{n-3} = \frac{\mu k}{\ell^2}}$$

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=\rho} = -\frac{nk}{\rho^{n+1}} + \frac{3\ell^2}{\mu\rho^4} > 0$$

$$\rightarrow -\frac{nk}{\rho^{n-3}} + \frac{3\ell^2}{\mu} > 0$$

$$\rightarrow (3-n)\frac{\ell^2}{\mu} > 0$$

því er ljóst að stöðug hringbraut fæst ef $\underline{n < 3}$

en er þetta nóg?

$$F(r) = -\mu g(r) = -\frac{\partial U}{\partial r}$$

$$\to \ddot{r} - r\dot{\theta}^2 = -g(r)$$

er hreyfijafnan, og með hverfiþunganum

$$\ddot{r} - \frac{\ell^2}{\mu^2 r^3} = -g(r)$$

Skoðum truflun um hringbraut
með geisla $\varrho$

$r \to \varrho + x$, $\varrho$ er fasti

$$\to \ddot{r} \to \ddot{x}$$

$$\to \ddot{x} - \frac{\ell^2}{\mu^2 \varrho^3 \left\{1 + \left(\frac{x}{\varrho}\right)\right\}^3} = -g(\varrho + x)$$

gerum ráð fyrir

$$\frac{x}{\varrho} << 1$$

$$\to \left\{1 + \left(\frac{x}{\varrho}\right)\right\}^{-3} \simeq 1 - 3\frac{x}{\varrho} + \cdots$$

$$g(\varrho + x) \simeq g(\varrho) + x g'(\varrho) + \cdots$$

því fæst

$$\ddot{x} - \frac{\ell^2}{\mu^2 \varrho^3}\left\{1 - \frac{3x}{\varrho}\right\} \simeq -\left\{g(\varrho) + x g'(\varrho)\right\}$$

í upphafi $\dot{r}\big|_{r=\varrho} = 0$

$$\ddot{r}\big|_{r=\varrho} = 0$$

$$g(\varrho) = \frac{\ell^2}{\mu^2 \varrho^3}$$

Jafnan verður því

$$\ddot{x} - g(\rho)\left\{1 - \frac{3x}{\rho}\right\} \simeq -\left\{g(\rho) + x\,g'(\rho)\right\}$$

$$\rightarrow \quad \boxed{\ddot{x} + \underbrace{\left\{\frac{3g(\rho)}{\rho} + g'(\rho)\right\}}\, x \simeq 0}$$

Skilgreinum

$$\omega_0^2 = \frac{3g(\rho)}{\rho} + g'(\rho)$$

$$\rightarrow \quad \ddot{x} + \omega_0^2 x = 0$$

með lausn

$$x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

Ef $\omega_0^2 < 0$

er lausnin með vaxandi og minnkandi lið

$\rightarrow$ óstöðug lausn

$\rightarrow$ Þarf því $\omega_0^2 > 0$

$$\rightarrow \quad \frac{3g(\rho)}{\rho} + g'(\rho) > 0$$

$$\rightarrow \quad \boxed{\frac{F'(\rho)}{F(\rho)} + \frac{3}{\rho} > 0}$$

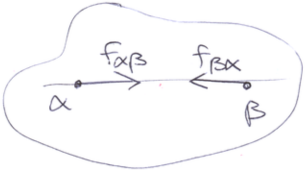
Setjum inn $F(r) = -\frac{k}{r^n}$

$$\rightarrow \quad (3-n)\frac{1}{\rho} > 0$$

svo eins og áður $n<3$

## Aflfræði agnakerfis

Innri kraftar agnakerfis




um krafta tveggja agna á hvor aðra gildir 3. lögmál Newtons

$$\bar{F}_{\alpha\beta} = -\bar{F}_{\beta\alpha}$$

veika útgáfan

og sterka útgáfan

kraftarnir liggja á tengilínu agnanna

sterkari útgáfan er ekki rétt fyrir t.d. rafsegulkrafta

⋮

## Massamiðja

Um heildarmassa kerfis gildir

$$M = \sum_{\alpha=1}^{n} m_\alpha$$

Eigum eftir að sjá að massamiðja

$$\bar{R} = \frac{1}{M}\sum_{\alpha=1}^{n} m_\alpha \bar{r}_\alpha$$

eða

$$\bar{R} = \frac{1}{M}\int \bar{r}\, dm$$

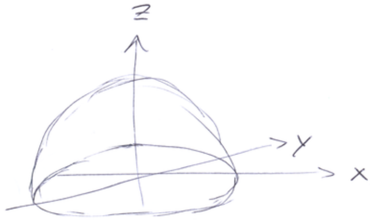
mun leika mikilvægt hlutverk.

## Dæmi

Finnum massamiðju gegnheils hálfkúluhvels með einsleitan þéttleika

$$\rho = \frac{M}{V} = \frac{M}{\frac{2\pi}{3}a^3}$$

þar sem $a$ er geisli hvelsins

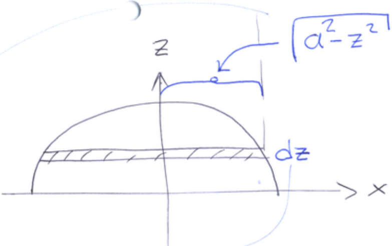



$$\bar{X} = \frac{1}{M}\int_{-a}^{a} x\, dm = 0$$ vegna samhverf

$$Y = \frac{1}{M}\int_{-a}^{a} y\,dm = 0$$

en

$$Z = \frac{1}{M}\int_{0}^{a} z\,dm$$




$$dm = \rho\, dV = \underbrace{\rho\pi(a^2 - z^2)dz}_{\text{„}\rho\pi r^2 h\text{"}}$$

$$\rightarrow Z = \frac{1}{M}\int_{0}^{a} z\rho\pi(a^2 - z^2)dz = \frac{\pi\rho a^4}{4M} = \frac{3}{8}a$$

---

**Skriðþungi**

Tökum ögn númer $\alpha$ í kerfinu
á hana verka ytri kraftar $\overline{F}_\alpha^{(e)}$
og innri kraftar

$$\overline{f}_\alpha = \sum_\beta \overline{f}_{\alpha\beta}$$

vegna hinna agnanna í kerfinu

Heildarkrafter á ögu $\alpha$ er þá

$$\bar{F}_\alpha = \bar{F}_\alpha^{(e)} + \bar{f}_\alpha$$

3. lögmál Newtons $\rightarrow \bar{f}_{\alpha\beta} = -\bar{f}_{\beta\alpha}$

2. lögmál Newtons

$$\dot{\bar{p}}_\alpha = m_\alpha \ddot{\bar{r}}_\alpha = \bar{F}_\alpha^{(e)} + \bar{f}_\alpha$$

$$\rightarrow \frac{d^2}{dt^2}(m_\alpha \bar{r}_\alpha) = \bar{F}_\alpha^{(e)} + \sum_\beta \bar{f}_{\alpha\beta}$$

Summum yfir $\alpha$

$$\frac{d^2}{dt^2} \sum_\alpha m_\alpha \bar{r}_\alpha = \sum_\alpha \bar{F}_\alpha^{(e)} + \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} \bar{f}_{\alpha\beta}$$

$$\frac{d^2}{dt^2}\sum_\alpha m_\alpha \bar{r}_\alpha = M\ddot{\bar{R}}$$

$$\sum_\alpha \bar{F}_\alpha^{(e)} \equiv \bar{F}, \qquad \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} f_{\alpha\beta} = \sum_{\alpha<\beta}\left\{\bar{f}_{\alpha\beta} + \bar{f}_{\beta\alpha}\right\} = 0$$

$$\rightarrow \boxed{M\ddot{\bar{R}} = \bar{F}}$$

Massamiðja kerfisins hreyfist eins og ögn með heildarkraftinn á hana

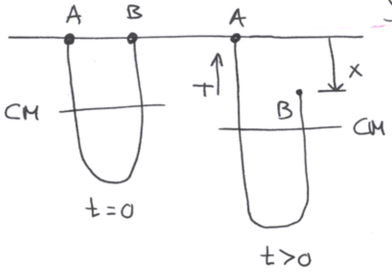
$$\bar{P} = \sum_\alpha m_\alpha \dot{\bar{r}}_\alpha = \frac{d}{dt}\left\{\sum_\alpha m_\alpha \bar{r}_\alpha\right\} = \frac{d}{dt}\left\{M\bar{R}\right\} = M\dot{\bar{R}}$$

og $$\dot{\bar{P}} = M\ddot{\bar{R}} = \bar{F}$$

Dæmi um keðju

lengd $b$, $M$

$\rho = \frac{M}{b}$




Reynum tvær aðferðir til að finna T

(1) $\dot{P} = Mg - T$

skoðum sem 1D hreyfingu

$$P = \rho\left[\frac{b-x}{2}\right]\dot{x}$$

$$\rightarrow \dot{P} = \frac{\rho}{2}\left\{-\dot{x}^2 + \ddot{x}(b-x)\right\}$$

Frjálst fall $\rightarrow$ $x = \frac{gt^2}{2}$

$$\rightarrow \dot{x} = gt = \sqrt{2gx}$$

$$\ddot{x} = g$$

$$\rightarrow \dot{P} = \frac{\rho}{2}\left\{-2gx + g(b-x)\right\}$$

$$= \frac{\rho}{2}\left\{gb - 3gx\right\} = Mg - T$$

$$\rightarrow T = Mg - \frac{\rho}{2}\left\{gb - 3gx\right\}$$

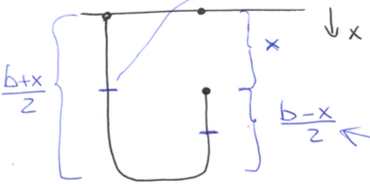
$$= Mg - \frac{M}{2b}\left\{gb - 3gx\right\}$$

$$= \frac{Mg}{2}\left\{\frac{3x}{b} + 1\right\}$$

② Notum orku, gerum ráð fyrir varðveislu

$$U(t=0) = U_0 = -\frac{\rho g b^2}{4} = \left\{-Mg\frac{b}{4}\right\}$$

Eftir fall um x




$$U = -g\rho\overset{\text{Massi}}{\left(\frac{b+x}{2}\right)}\overset{\text{hæðmiðju}}{\left(\frac{b+x}{4}\right)}$$

$$-g\rho\underbrace{\left(\frac{b-x}{2}\right)}_{\text{massi}}\underbrace{\left(x+\frac{b-x}{4}\right)}_{\text{hæð}}$$

$$U = -g\rho\left\{\frac{b^2+2bx+x^2}{2\cdot 4} + \frac{b^2-2bx+x^2}{2\cdot 4} + \frac{bx-x^2}{2}\right\}$$

$$= -\frac{1}{4}g\rho\left\{b^2 + 2bx - x^2\right\}$$

Hreyfiorka hogri hlutans í falli

$$K = \frac{\rho}{4}(b-x)\dot{x}^2$$

Orkan er ~~varðveitt~~

$$K + U = \frac{\rho}{4}(b-x)\dot{x}^2 - \frac{1}{4}g\rho\left\{b^2 + 2bx - x^2\right\} = U_0 = -\frac{\rho g}{4}b^2$$

$$\rightarrow \dot{x}^2 = \frac{g(2bx - x^2)}{b-x} \quad \rightarrow 2\dot{x}\ddot{x} = \frac{g(2b\dot{x} - 2x\dot{x})}{b-x} + \frac{g(2bx-x^2)}{(b-x)^2}\dot{x}$$

$$\rightarrow \ddot{x} = g + \frac{g(2bx - x^2)}{2(b-x)^2}$$

notum í

$$\dot{P} = \frac{\rho}{2}\left\{-\dot{x}^2 + \ddot{x}(b-x)\right\} = Mg - T$$

$$\rightarrow \boxed{T = \frac{Mg}{4b}\frac{1}{(b-x)}\left\{2b^2 + 2bx - 3x^2\right\}} \quad (2)$$

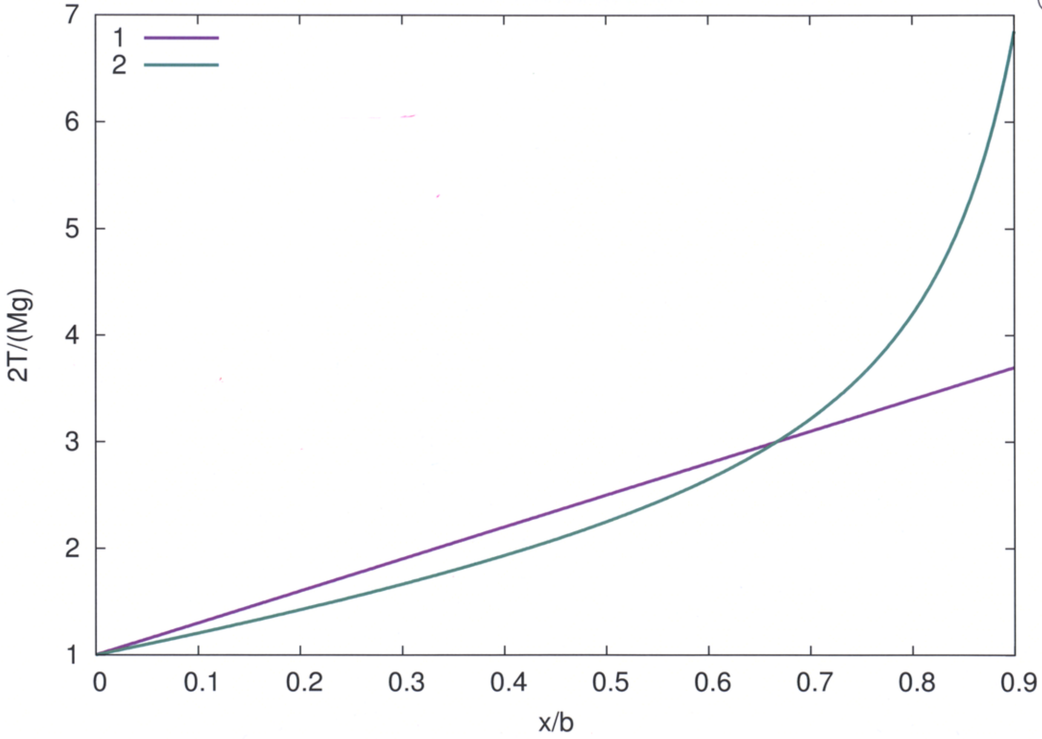
Þó nokkuð ólíkt fyrri niðurstöðu

$$T = \frac{Mg}{2}\left\{\frac{3x}{b} + 1\right\} \quad (1)$$

Athugum á grafi

$$\frac{Mg}{2}\left\{\frac{b^2}{b(b-x)}\left[1 + \frac{x}{b} - \frac{3}{2}\left(\frac{x}{b}\right)^2\right]\right\}$$

$$\frac{1}{1-\frac{x}{b}}$$


1
2
2T/(Mg)
x/b
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
3
4
5
6
7