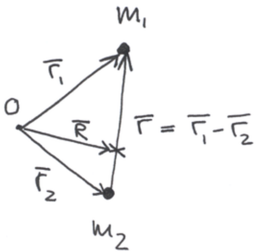
Hreyfing í miðlægum krafti



Tveir massar – skertur massi

tveir massar $\rightarrow$ 3 hnit $\times 2$
$\rightarrow$ 6 breytur

Þægilegt að nota massamiðjuhnit $\bar{R}$ og innbyrðishnit

$$\bar{r} \equiv \bar{r}_1 - \bar{r}_2$$

Mættið er aðeins fall af

$$r = |\bar{r}_1 - \bar{r}_2|$$

$$L = \frac{1}{2}\left\{ m_1 |\dot{\bar{r}}_1|^2 + m_2 |\dot{\bar{r}}_2|^2 \right\} - U(r)$$

Ef við höfum ekki áhuga á hreyfingu $\bar{R}$ (CM) notum

$$m_1 \bar{r}_1 + m_2 \bar{r}_2 = 0$$

$$\bar{r} = \bar{r}_1 - \bar{r}_2$$

$$\begin{pmatrix} m_1 & m_2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{r}_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{r} \end{pmatrix}$$

$$\rightarrow \quad \bar{r}_1 = \frac{m_2}{m_1+m_2}\bar{r}$$

$$\bar{r}_2 = -\frac{m_1}{m_1+m_2}\bar{r}$$

notum í

$$T = \frac{1}{2}\left\{ m_1|\dot{\bar{r}}_1|^2 + m_2|\dot{\bar{r}}_2|^2 \right\}$$

$$= \frac{1}{2}\left\{ \frac{m_1 m_2^2}{(m_1+m_2)^2} + \frac{m_2 m_1^2}{(m_1+m_2)^2} \right\} |\dot{\bar{r}}|^2$$

$$= \frac{1}{2}\left\{ \frac{(m_1+m_2)m_1 m_2}{(m_1+m_2)^2} \right\} |\dot{\bar{r}}|^2$$

$$= \frac{1}{2}\left\{ \frac{m_1 m_2}{(m_1+m_2)} \right\} |\dot{\bar{r}}|^2 = \frac{1}{2}\mu|\dot{\bar{r}}|^2$$

ef $\boxed{\mu \equiv \frac{m_1 m_2}{m_1+m_2}}$

μ er kallaður skertur massi (e. reduced mass)

því verður nú

$$\boxed{L = \frac{1}{2}\mu|\dot{\bar{r}}|^2 - U(r)}$$

↑

einungis umbyrðis hnit $\bar{r}$ kemur fyrir

varðveisla - fyrsti heildisfasti hreyfingarinnar
(e. first integral of motion)

vegna þess að $U = U(r)$ með lengd en ekki stefnu

$\rightarrow$ $\boxed{\text{snúningur um ás í gegnum massamiðju breytir engu}}$

$\bar{L} = \bar{r} \times \bar{p} = \text{fasti}$ $\rightarrow$ hreyfing í sléttu því $\bar{r}$ og $\bar{p}$ verða að liggja í sömu sléttu þvert á $\bar{L}$

hverfiþungi

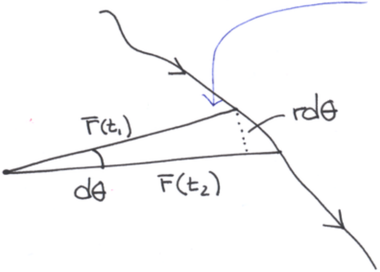
$$\rightarrow \boxed{L = \frac{1}{2}\mu\left\{\dot{r}^2 + (r\dot{\theta})^2\right\} - U(r)}$$ $\leftarrow$ lagrange fall

$$\rightarrow \dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \rightarrow p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{fasti}$$

p_θ er fyrsti heildisfastinn, gefum nafn

$$\boxed{l \equiv \mu r^2 \dot{\theta} = \text{fasti}}$$

en



flötur $dA = \overbrace{\frac{1}{2} r^2 d\theta}^{\frac{1}{2} r \cdot r d\theta}$

$\rightarrow$ flatar ferð

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{r^2}{2}\dot{\theta}$$

$$= \frac{l}{2\mu} = \text{fasti}$$

Annað lögmál Keplers

$\boxed{\text{Ekki vegna } \frac{1}{r^2} \text{ krafts}}$

Enginn viðnámskraftur

$\to \quad T + U = E = \text{fasti}$

$$\to \quad \boxed{E = \frac{\mu}{2}\left\{ \dot{r}^2 + (r\dot{\theta})^2 \right\} + U(r) = \frac{\mu}{2}\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)}$$

sem við getum nýtt í hreyfijöfnu

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - U(r)\right) - \frac{\ell^2}{\mu^2 r^2}}$$

$$\to \quad d\theta = \frac{\pm \frac{\ell}{r^2} dr}{\mu\sqrt{\frac{2}{\mu}\left(E - U(r)\right) - \frac{\ell^2}{\mu^2 r^2}}}$$

notum

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$$

notum $\dot{\theta} = \frac{\ell}{\mu r^2}$

$$\to d\theta = \frac{\ell}{\mu r^2}\frac{dr}{\dot{r}}$$

$$\theta(r) = \int \frac{\pm \frac{l}{r^2} dr}{\sqrt{2\mu\left[E - U(r) - \frac{l^2}{2\mu r^2}\right]}}$$

l : fasti
→ $\dot{\theta}$ hefur alltaf sama formerkið
→ θ vex eða minnkar einhalla með tíma

Formleg lausn, nákvæm greinilausn er aðeins þekkt fyrir nokkur sértilfelli

T.d. fyrir $F(r) \sim \frac{1}{r^n}$ eru lausnir þekktar í elliptískum heildum og föllum. $n = 1, -2, -3$ gefur lausnir í hornaföllum

H.O. og $\frac{1}{r^2}$

Má nýta til að skoða samhverfur og vissa eiginleika

Ekki þægilegt form fyrir tölulega reikninga. Þar er hreyfijafnan á afleiðu formi betri

Skoðum

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = 0$$

með

$$L = \frac{\mu}{2}\left\{\dot r^2 + (r\dot\theta)^2\right\} - U(r)$$

þá fæst hreyfijafnan

$$\mu\left\{\ddot r - r\dot\theta^2\right\} = -\frac{\partial U}{\partial r} = F(r)$$

sem er heppileg til tölulegra reikninga með $\dot\theta = \frac{\ell}{\mu r^2}$

En, skoðum brautarskipti

$$u \equiv \frac{1}{r}$$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$= -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot r}{\dot\theta}$$

$$\rightarrow \frac{du}{d\theta} = -\frac{\mu}{\ell}\dot r$$

ef notað er $\dot\theta = \frac{\ell}{\mu r^2}$

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\mu}{\ell}\dot r\right)$$

$$= \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{\ell}\dot r\right) = -\frac{\mu\ddot r}{\ell\dot\theta}$$

$$= -\frac{\mu^2}{\ell^2}r^2\ddot r$$

Munitum því

$$\ddot{r} = -\frac{\ell^2}{\mu^2} u^2 \frac{d^2u}{d\theta^2}$$

$$r\dot{\theta}^2 = \frac{\ell^2}{\mu^2} u^3$$

$$\underbrace{\mu\left[\ddot{r} - r\dot{\theta}^2\right] = F(r)}$$

$$\Downarrow$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2 u^2} F(1/u)$$

$$\Downarrow$$

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2} F(r)}$$

Dæmi

Þekkjum braut $r(\theta)$ hvaða kraftur veldur henni?

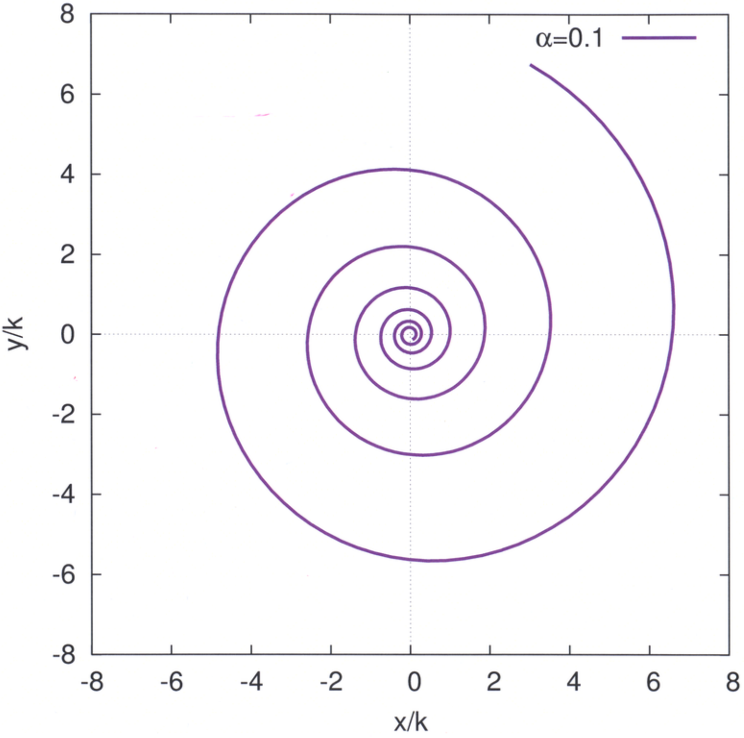
$$r(\theta) = k e^{\alpha\theta}$$

Logragormur, sjá mynd á næstu síðu

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{e^{-\alpha\theta}}{k}\right) = -\frac{\alpha e^{-\alpha\theta}}{k}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2 e^{-\alpha\theta}}{k} = \frac{\alpha^2}{r}$$

$$\rightarrow \boxed{F = -\frac{\ell^2}{\mu r^2}\left[\frac{\alpha^2}{r} + \frac{1}{r}\right] = -\frac{\ell^2}{\mu r^3}\left\{\alpha^2 + 1\right\}}$$



$r(\theta) = k e^{\alpha \theta}$

Hvernig er brautin háð tíma?

$$\dot{\theta} = \frac{l}{\mu r^2} = \frac{l}{\mu k^2 e^{2\alpha\theta}} \longrightarrow e^{2\alpha\theta} d\theta = \frac{l}{\mu k^2} dt$$

heildum

heildunarfasti

$$\frac{e^{2\alpha\theta}}{2\alpha} = \frac{lt}{\mu k^2} + \frac{C}{2\alpha} \longrightarrow e^{2\alpha\theta} = \frac{2\alpha l t}{\mu k^2} + C$$

eða

$$\theta(t) = \frac{1}{2\alpha} \ln\left\{\frac{2\alpha l t}{\mu k^2} + C\right\} \qquad r(\theta) = k e^{\alpha\theta}$$

$$r(t) = \sqrt{\frac{2\alpha l}{\mu} t + k^2 C}$$

$$U(r) = -\frac{l^2(\alpha^2+1)}{2\mu r^2} \quad \text{ef} \quad U(r) \xrightarrow[r\to\infty]{} 0$$

þá fæst $E = 0$ sem kemur ekki á óvart þegar brautin er stöðuð, massinn er ekki bundinn,

Brautir í miðlægu mætti

fyrir útþátt hraðans fékkst

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}}$$

→ $\dot{r} = 0$ þegar

$$E - U(r) - \frac{l^2}{2\mu r^2} = 0$$

Viðsnúningspunktar

venjulega tvær rætur

→ r_{max} og r_{min}

Ef ein rót → hringbraut

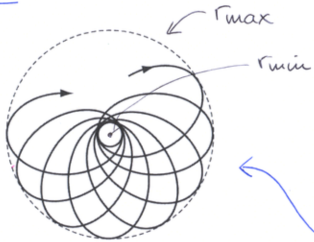


Fig 8-4 Thornton/Marion

Ef hreyfingin er lotubundin þá er talað um lokaða braut

Ef ekki, fæst opin braut

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{l\,dr}{r^2\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}$$

ef $\Delta\theta$ er $2\pi\left(\frac{n}{m}\right)$ með n, m heiltölum

→ lokuð braut

fyrir $U(r) \sim r^{n+1}$ fást lokaðar ekki hringlaga brautir fyrir $n = -2$ og $+1$

Miðflótta....

Í stofninum $\sqrt{E-U-\frac{l^2}{2\mu r^2}}$ er síðasti liðurinn með vídd orku

$$\frac{l^2}{2\mu r^2}=\frac{1}{2}\mu r^2\dot{\theta}^2$$

Ef við lítum á þetta sem hluta af „stöðuorku"

$$U_c\equiv\frac{l^2}{2\mu r^2}$$

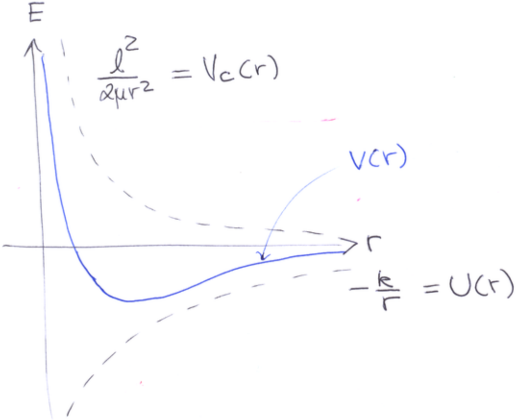
þá fæst „krafturinn"

$$F_c=-\frac{\partial U_c}{\partial r}=\frac{l^2}{\mu r^3}=\mu r\dot{\theta}^2$$

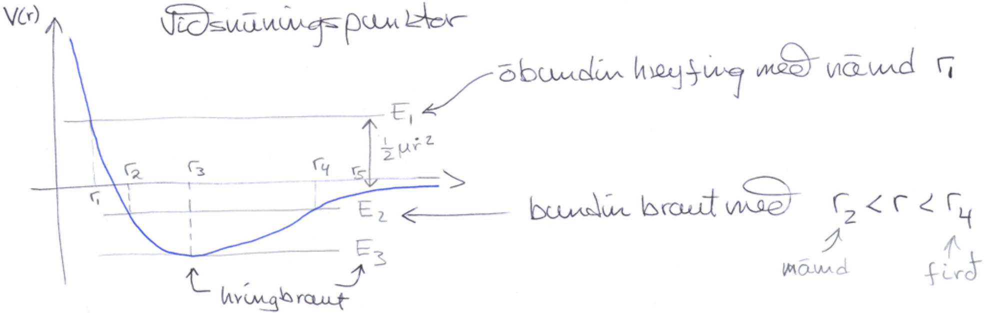
gervikraftur – miðflóttakraftur, þá verður virka stöðuorkan

$$V(r)\equiv U(r)+\frac{l^2}{2\mu r^2}$$

$\boxed{\text{vex með hverfiþunga}}$



E
$\frac{l^2}{2\mu r^2} = V_c(r)$
V(r)
r
$-\frac{k}{r} = U(r)$



V(r)
Viðsnúningspunktar
óbundin hreyfing með nánd r_1
E_1
$\frac{1}{2}\mu\dot{r}^2$
r_1
r_2
r_3
r_4
r_5
E_2
bundin braut með $r_2 < r < r_4$
nánd
firð
E_3
hringbraut