Um Hreyfiorku

Mummtengsl hnita við alhnit

$$x_{\alpha,i} = x_{\alpha,i}(q_j, t) \qquad \begin{array}{l} \alpha = 1,2,\dots n \\ i = 1,2,3 \\ j = 1,2,\dots s \end{array}$$

Í föstum Kartísumhnitum

er hreyfiorkan

$$T = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha,i}^2$$

$$\dot{x}_{\alpha,i} = \sum_{j=1}^{s} \frac{\partial x_{\alpha,i}}{\partial q_j} \dot{q}_j + \frac{\partial x_{\alpha,i}}{\partial t}$$

$$\rightarrow \quad \dot{x}_{\alpha,i}^2 = \sum_{j,k} \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial q_k} \dot{q}_j \dot{q}_k + 2 \sum_j \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial t} \dot{q}_j + \left(\frac{\partial x_{\alpha,i}}{\partial t} \right)^2$$

Því verður hreyfiorkan

$$T = \sum_{\alpha} \sum_{ijk} \left[\frac{m_\alpha}{2} \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial q_k} \dot{q}_j \dot{q}_k \right] + \sum_{\alpha} \sum_{i,j} \left[m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial t} \dot{q}_j \right]$$

$$+ \sum_{\alpha} \sum_{i} \left[\frac{m_\alpha}{2} \left(\frac{\partial x_{\alpha,i}}{\partial t} \right)^2 \right]$$

Sem hefur formið

$$T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k + \sum_j b_j \dot{q}_j + c$$

engin tími

Ef kerfið er stjarfnefnt (scleronomic) svo $x_{\alpha,i} = x_{\alpha,i}(q_j)$

þá er $\frac{\partial x_{\alpha,i}}{\partial t} = 0$ og $b_j = 0$, $c = 0$

$$\rightarrow \boxed{T = \sum_{jk} a_{jk} \dot{q}_j \dot{q}_k}$$

óhliðruð annarsstigs jafna af ferðinni í alhnitum skoðum

$$\frac{\partial T}{\partial \dot{q}_l} = \sum_k a_{lk} \dot{q}_k + \sum_j a_{jl} \dot{q}_j$$

$$\rightarrow \sum_l \dot{q}_l \frac{\partial T}{\partial \dot{q}_l} = \sum_{kl} a_{lk} \dot{q}_k \dot{q}_l + \sum_{jl} a_{jl} \dot{q}_j \dot{q}_l$$

og því

$$\boxed{\sum_l \dot{q}_l \frac{\partial T}{\partial \dot{q}_l} = 2T}$$

notum rétt bráðum

sértilfelli

Setning Eulers

Ef $f(y_k)$ er óhliðrað fall af y_k af gráðu n

$$\rightarrow \sum_k y_k \frac{\partial f}{\partial y_k} = n f$$

Varðveislulögmál

Ef tíminn er allsstaðar einsleitur

$$\frac{\partial L}{\partial t} = 0 \quad \longrightarrow \quad \frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$$

notum jöfnur Lagrange

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad \longrightarrow \quad \frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$$

og því

$$\frac{dL}{dt} = \sum_j \left\{ \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \right\} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)$$

$$\longrightarrow \quad \frac{d}{dt}\left\{ L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right\} = 0$$

fasti í tíma

Köllum

$$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \text{fasti}$$

Veljum

$$U = U(x_{\alpha,i}) \quad \text{óháð } \dot{x}_{\alpha,i}$$

höfum valið

$$x_{\alpha,i} = x_{\alpha,i}(q_j)$$
$$q_j = q_j(x_{\alpha,i})$$

óháð tíma

$$\longrightarrow U = U(q_j) \text{ og } \frac{\partial U}{\partial \dot{q}_j} = 0 \longrightarrow \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

$$\boxed{H = \underbrace{\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}}_{2T} - \{T-U\} = T+U = \text{fasti}}$$

Fall Hamiltons

ummyndanir háðar óháð tíma
U er óháð hraða

skriðþungi

Rúmið er einsleitt í tregðukerfi

→ L fyrir lokað kerfi er óbreytt

ef rúminu er hnikað $\bar{r}_\alpha \to \bar{r}_\alpha + \delta\bar{r}$

Notum Kartísk hnit

hér $L = L(x_i, \dot{x}_i)$

$$\delta\bar{r} = \sum_i \hat{e}_i \delta x_i$$

↑ algerlega óháð t

$$\delta\dot{x}_i = \delta\left(\frac{dx_i}{dt}\right) = \frac{d}{dt}\left(\delta x_i\right) \equiv 0$$

$$\to \delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i = 0$$

óháðar hnikanir δx_i

$$\to \frac{\partial L}{\partial x_i} = 0 \quad \to \text{Lagrange jöfnur} \to \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$$

$$\to \frac{\partial L}{\partial \dot{x}_i} = \text{fasti} = \frac{\partial(T-U)}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\left\{\frac{1}{2}m\sum_j \dot{x}_j^2\right\} = m\dot{x}_i = p_i$$

$$= \text{fasti}$$

Við sjáum því að

$$\boxed{p_i = \frac{\partial L}{\partial \dot{x}_i}}$$

Í þessum hnitum er jöfnur Lagrange

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$$

$$\longrightarrow \boxed{\dot{p}_i = \frac{\partial L}{\partial x_i}}$$

Í kartískum hnitum eru þessar jöfnur jafngildar jöfnum Lagrange

Í einsleitu rúmi, má með örsmárri snúningi

$$\delta \bar{r} = \delta \hat{\theta} \times \hat{r}$$

sýna að

$$\bar{r} \times \bar{p} = \text{fasti}$$

Hverfiþunginn er varðveittur

Tregðukerfi	Tími einsleitur	rúm einsleitt	rúm einsleitt
L	óháð t	óháð örsmárri færslu	óháð örsmárri snúning
Varðveitt	Heildarorka	skriðþungi	hverfiþungi

Emmy Noether

Jöfnur Hamiltons

Í Kartískum hnitum höfum við

$$p_i = \frac{\partial L}{\partial \dot{x}_i}$$

Við víkkum þetta út fyrir alhnit

$$\boxed{p_j \equiv \frac{\partial L}{\partial \dot{q}_j}}$$

Jöfnur Lagrange verða því

$$\boxed{\dot{p}_j = \frac{\partial L}{\partial q_j}}$$

Samtímis verður

$$\boxed{\begin{aligned} H &= \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \\ &= \sum_j p_j \dot{q}_j - L \end{aligned}}$$

notum til að finna $\dot{q}_j = \dot{q}(q_k, p_k, t)$

þá sést að

$$\boxed{H(q_k, p_k, t) = \sum_j p_j \dot{q}_j - L(q_k, \dot{q}_k, t)}$$

(*)

hnitaskipti (Legrende ummyndun)

$$\rightarrow dH = \sum_k \left\{ \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k \right\} + \frac{\partial H}{\partial t} dt$$

En samkvæmt (*)

$$dH = \sum_k \left\{ \dot{q}_k dp_k + p_k d\dot{q}_k - \overbrace{\frac{\partial L}{\partial q_k}}^{=\dot{p}_k} dq_k - \overbrace{\frac{\partial L}{\partial \dot{q}_k}}^{=p_k} d\dot{q}_k \right\} - \frac{\partial L}{\partial t} dt$$

$$= \sum_k \left\{ \dot{q}_k dp_k - \dot{p}_k dq_k \right\} - \frac{\partial L}{\partial t} dt$$

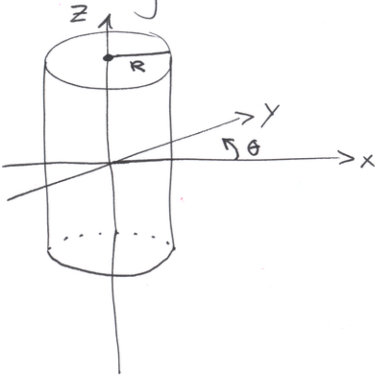
$$\longrightarrow \boxed{\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial H}{\partial q_k} \end{aligned}} \quad \text{og} \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad \text{og} \quad \frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Hreyfijöfnur Hamiltons

Körjöfnur hreyfingar

Skoðum tvö dæmi

(1)
"Ögn hreyfist á yfirborði sívalnings"



Í mætti $U = \frac{1}{2} k r^2$

$$r^2 = x^2 + y^2 + z^2 = R^2 + z^2$$

$$v^2 = \overbrace{\dot{R}^2}^{=0} + (R\dot{\theta})^2 + \dot{z}^2 = (R\dot{\theta})^2 + \dot{z}^2$$

$$\rightarrow T = \frac{m}{2}\left\{(R\dot{\theta})^2 + \dot{z}^2\right\}$$

$$L = T - U = \frac{m}{2}\left\{(R\dot{\theta})^2 + \dot{z}^2\right\} - \frac{k}{2}\left\{R^2 + z^2\right\}$$

alhnitin eru θ og z og alskriðþungarnir eru

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

Kerfið er lokað, geymið og hnitaskiptin eru óháð tíma

$$H(z, p_\theta, p_z) = T + U = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}kz^2$$

Liðnum $\frac{1}{2}kR^2$ er sleppt, því hann er fasti

Reynum hreyfijöfnur Hamiltons

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

$$\rightarrow \boxed{p_\theta = mR^2\dot{\theta} = \text{fasti}}$$

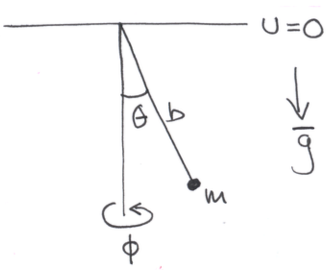
Hverfiþunginn um z-ás er fasti

$$\rightarrow m\ddot{z} + kz = 0$$

$$\rightarrow \boxed{\ddot{z} + \frac{k}{m}z = 0}$$

ω_0^2

2 Kúlupendúll



Alhnitin eru θ og ϕ

$$T = \frac{m}{2}\left\{(b\dot{\theta})^2 + (b\sin\theta\,\dot{\phi})^2\right\}$$

$$U = -mgb\cos\theta$$

Alskriðþungar

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mb^2\sin^2\theta \cdot \dot{\phi}$$

$$\rightarrow \dot{\theta} = \frac{p_\theta}{mb^2} \quad \text{og} \quad \dot{\phi} = \frac{p_\phi}{mb^2\sin^2\theta}$$

notum í T

$$\boxed{\begin{aligned} H &= T + U \\ &= \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} - mgb\cos\theta \end{aligned}}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mb^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial \phi} = \frac{p_\phi}{mb^2 \sin^2\theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{mb^2 \sin^3\theta} - mgb\sin\theta$$

$$\dot{p}_\phi = \frac{\partial H}{\partial \phi} = 0$$

ϕ kemur ekki fyrir í H

$\to$ p_ϕ er fasti um samhverfuásinn,

hverfiþungi pendúlsins er varðveittur

höfðum $\dot{\theta} = \frac{p_\theta}{mb^2}$

$\to$ $\dot{p}_\theta = \ddot{\theta}\, mb^2$

því fæst

$$\ddot{\theta} - \frac{p_\phi^2 \cos\theta}{(mb^2)^2 \sin^3\theta} + \frac{g}{b}\sin\theta = 0$$

Sem mætti líka finna með jöfnum Lagrange

eða

lítum á jöfnur Hamiltons sem fyrsta stigs jöfnukerfi . . .