Lögmál Hamiltons – Aflfræði Lagrange

Af öllum mögulegum leiðum fyrir kerfi frá einum punkti til annars á ákveðnu tímabili (með skorðum), er raunverulega leiðin sú sem lágmarkar tímaheildi munar hreyfi- og stöðuorku

$$\boxed{\delta \int_{t_1}^{t_2} \{T - U\} dt = 0}$$

Ef $T = T(\dot{x}_i)$, $U = U(x_i)$

$$L \equiv T - U = L(x_i, \dot{x}_i)$$

$$\rightarrow \delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i) dt = 0$$

Við notum því vitneskju úr síðasta fyrirlestri, hnitum og sjáum að hreyfijöfnurnar fást með

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0}$$

Hreyfijafna Lagrange

L : fall Lagrange

* skoðum dæmi ← t.þ.a sjá að þetta verkar
* kynnum alhnit
* skoðum flókin dæmi

generalized coordinates

sjáum að jöfnur Lagrange með alhnitum einfalda viðfangsefni aflfræðinnar fyrir okkur

Hreintónasveifill

$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{\partial L}{\partial x} = -kx \, , \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

og $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\rightarrow \; -kx - m\ddot{x} = 0$$

$$\rightarrow \; \boxed{m\ddot{x} + kx = 0}$$

þekkt niðurstaða

Pendúll

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta \, , \quad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\rightarrow \; -mgl\sin\theta - ml^2\ddot{\theta} = 0$$

$$\rightarrow \; \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

þurftum ekki að athuga $\dot{\hat{e}}_\theta$, θ var meðhöndlað eins og x!

Enginn Kraftur með flóknum vigurstrúktúr!
Einungis unnið með skalarstærðir!

Generaliseraðar — Alhnit

Hugsum okkur n-agnir → 3n hnit í 3D-rúmi

m-af þessum hnitum eru ekki óháð, einhverjar agnir gætu verið fastar saman, eða það eru skorður við braut

→ $s = 3n - m$ óháð hnit (frelsisgráður)

m - skorður

Veljum einhver s - óháð hnit, ekki til einkvæmt val

→ q_i alhnit ← þurfa ekki sömu vídd, $x, \theta, \ldots$

→ alhraðar $\dot{q}_i$ skalarstærðir

$$q_j = q_j(x_{\alpha,i}, t)$$
$$\dot{q}_j = \dot{q}_j(x_{\alpha,i}, \dot{x}_{\alpha,i}, t)$$

$\alpha = 1, 2, \dots n$

$i = 1, 2, 3 \quad (x, y, z)$

$j = 1, 2, 3, \dots S$

$S = 3n - m$

$+$ skorður $\quad f_k(x_{\alpha,i}, t) = 0 \qquad k = 1, 2, \dots, m$

Dæmi um hnit

Eind á hálfhveli $\qquad x^2 + y^2 + z^2 - R^2 = 0, \quad z \geq 0$

Gætum reynt $\quad q_1 = \frac{x}{R}, \quad q_2 = \frac{y}{R}, \quad q_3 = \frac{z}{R}$

$$\rightarrow \quad q_1^2 + q_2^2 + q_3^2 = 1 \quad \leftarrow \text{ ekki óháð}$$

Veljum þá t.d. q_1 og q_3 og eina jöfnu fyrir skorður

$$z = \sqrt{R^2 - x^2 - y^2}$$

enda hreyfing á 2D-fleti

$\boxed{\text{Fyrir 1D pendúl eru } x \text{ og } y \text{ háð} \rightarrow \text{ alhnit } \theta}$

Lagrange í alhnitum

L er stakarfall + $L = T - U$

→ L er óbreytt við hnitaskipti

{Til eru umngjörðir á L sem óbreyta hreyfijöfnunum}

Breyting á núllpunkti U breytir ekki hreyfijöfnu

$$L = L(q_j, \dot{q}_j, t)$$

$$\delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt = 0 \qquad \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

$j = 1, 2, \ldots s$

* Við krefjunst (ekki neyðsyn, en annars part útvíkkan) að allir kraftar, nema støddur sum geymnir

* Krefjunst støða $f_k(x_{\alpha,i},t)=0$

heilnevndar støddur
holonomic

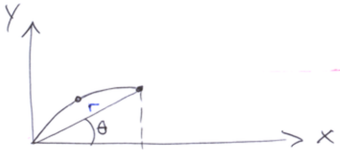
Ef $f_k(x_{\alpha,i})=0$

fastar støddur (fixed)
stjarfnevndar (scleronomic)

annars með t er þar flótinevndar (rheonomic)

skóðumdømi

Fallhreyfing, 2D, í tveimur hnitakerfum



$$T = \frac{1}{2} m \left\{ \dot{x}^2 + \dot{y}^2 \right\}$$

$$U = mgy$$

$$L = T - U = \frac{1}{2} m \left\{ \dot{x}^2 + \dot{y}^2 \right\} - mgy$$

x:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \quad \rightarrow \quad 0 - \ddot{x} = 0 \quad \rightarrow \quad \underline{\ddot{x} = 0}$$

y:

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\dot{y}}\right) = 0 \quad \rightarrow \quad -mg - m\ddot{y} = 0$$

$$\rightarrow \quad \underline{\ddot{y} + g = 0}$$

En í pólhnitum?

Rifjum upp $\bar{v} = \dot{\bar{r}} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$$\rightarrow v^2 = \bar{v}\cdot\bar{v} = \dot{r}^2 + (r\dot{\theta})^2$$

því fæst

$$L = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2\right] - mgr\sin\theta$$

r:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \rightarrow +mr\dot{\theta}^2 - mg\sin\theta - m\ddot{r} = 0$$

$$\rightarrow \boxed{\ddot{r} - r\dot{\theta}^2 + g\sin\theta = 0}$$

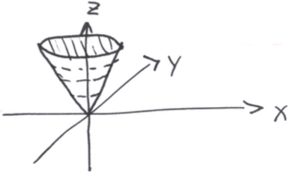
θ:

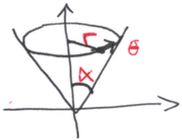
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \rightarrow -mgr\cos\theta - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} = 0$$

Hér er greinilegt að það er þægilegra að nota x og y sem alhnit

$$\rightarrow \boxed{r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + mgr\cos\theta = 0}$$

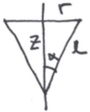
Dæmi Ögn hreyfist í þyngdarsviði inni í keiluyfirborði





Eðlilegt að nota sívalningshnit r, θ, z sem alhnit

Við höfum skorður



$$\left.\begin{aligned} l\cos\alpha &= z \\ l\sin\alpha &= r \end{aligned}\right\} \to \frac{z}{r} = \cot\alpha \quad \text{eða} \quad z = r\cot\alpha$$

(1.101) $$\bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z$$

$$\to v^2 = \dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2 = \dot{r}^2 + (r\dot{\theta})^2 + \dot{r}^2\cot^2\alpha$$

$$= \dot{r}^2\left\{1 + \cot^2\alpha\right\} + (r\dot{\theta})^2 = \dot{r}^2\frac{1}{\sin^2\alpha} + (r\dot{\theta})^2$$

Hér sést að $\alpha = \frac{\pi}{2}$ gefur einmitt 2D-kerfi í sléttu skorðurnar hafa skilið eftir tvö alhnit, enda hreyfing á yfirborði, eða í

$$U = mgz = mgr\cot\alpha$$

$$\rightarrow L = \frac{1}{2}m\left\{\dot{r}^2\frac{1}{\sin^2\alpha} + (r\dot{\theta})^2\right\} - mgr\cot\alpha$$

$\underline{\theta}$:

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = 0 \quad \rightarrow \quad 0 - \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

$$\rightarrow \boxed{mr^2\dot{\theta} = C \leftarrow \text{fasti}}$$

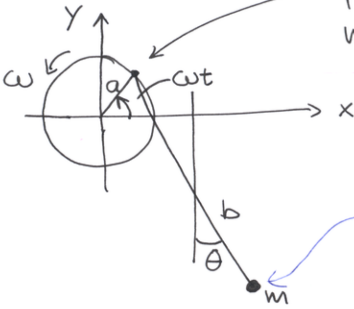
hverfiþungi agnarinnar um z-ás

$\underline{r}$:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) = 0 \quad \rightarrow \quad +r\dot{\theta}^2 m - mg\cot\alpha - m\frac{\ddot{r}}{\sin^2\alpha} = 0$$

$$\rightarrow \boxed{\ddot{r} - r\dot{\theta}^2\sin^2\alpha + g\frac{1}{2}\sin(2\alpha) = 0}$$

Dæmi



pendúll festur við jaðar hjóls sem snýst með jafnri hornferð ω

{ Hver vill reyna að finna krafta hér? }

$$x = a\cos(\omega t) + b\sin\theta$$

$$y = a\sin(\omega t) - b\cos\theta$$

$$\rightarrow \quad \dot{x} = -a\omega\sin(\omega t) + b\dot{\theta}\cos\theta$$

$$\dot{y} = a\omega\cos(\omega t) + b\dot{\theta}\sin\theta$$

$$\ddot{x} = -a\omega^2\cos(\omega t) + b\left\{\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right\}$$

$$\ddot{y} = -a\omega^2\sin(\omega t) + b\left\{\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right\}$$

því er θ eina alhnitið hér

$$L = T - U = \frac{1}{2} m \left\{ \dot{x}^2 + \dot{y}^2 \right\} - mgy$$

$$= \frac{m}{2} \left\{ a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega \left[-\sin(\omega t)\cos\theta + \cos(\omega t)\sin\theta \right] \right\}$$

$$- mg \left\{ a\sin(\omega t) - b\cos\theta \right\}$$

Ef $\omega = 0$ fáum við $\ddot{\theta} + \frac{g}{b}\sin\theta = 0$

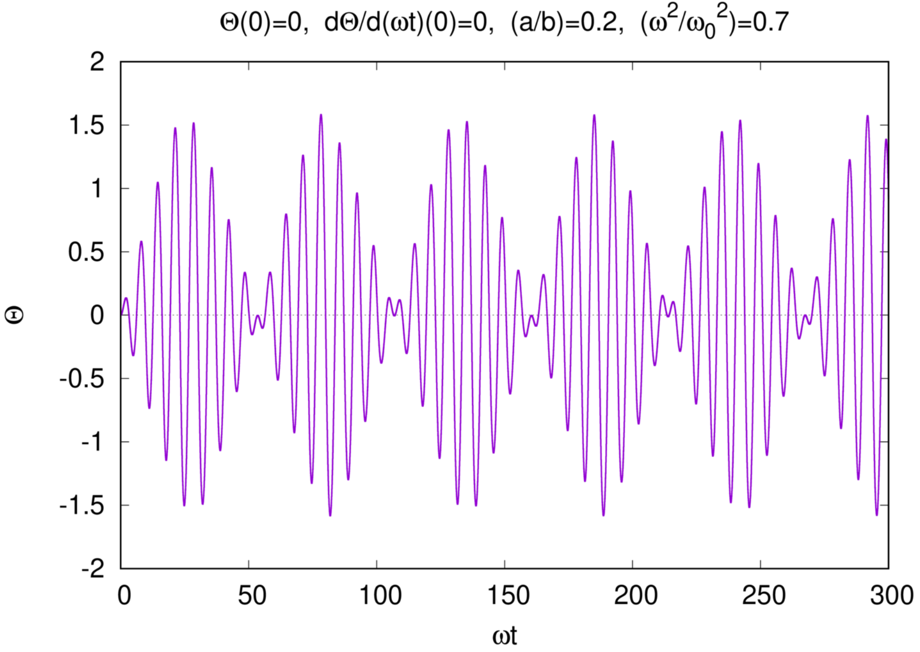
$$\rightarrow L = \frac{m}{2} \left\{ a^2\omega^2 + (b\dot{\theta})^2 + 2b\dot{\theta}a\omega \sin(\theta - \omega t) \right\}$$

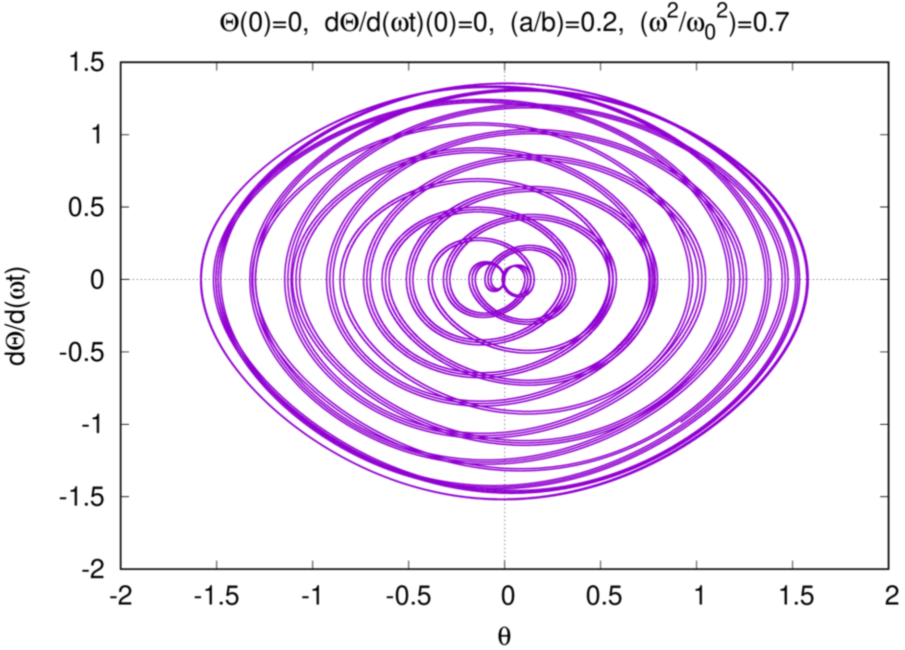
$$- mg \left\{ a\sin(\omega t) - b\cos\theta \right\}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0 \rightarrow mb\dot{\theta}a\omega\cos(\theta - \omega t) - mgb\sin\theta$$

$$- mb^2\ddot{\theta} - mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t)$$

$$\rightarrow \boxed{\ddot{\theta} - \frac{\omega^2 a}{b}\cos(\theta - \omega t) + \frac{g}{b}\sin\theta = 0}$$


Θ(0)=0, dΘ/d(ωt)(0)=0, (a/b)=0.2, (ω^2/ω_0^2)=0.7
Θ
ωt
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
50
100
150
200
250
300



Θ(0)=0, dΘ/d(ωt)(0)=0, (a/b)=0.2, (ω²/ω₀²)=0.7
dΘ/d(ωt)
θ
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2