Hnikun – Hnikareikningur

feikilega vítt svið .... löng saga ..... tölur. . . . . .

Við höfum áhuga á verkefnum þar sem t.d. spurt er hvaða fall y(x) gefur heildinu

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\} dx$$

útgildi? lág- eða hágildi með mörkin x_1 og x_2 venjulega föst

J : felli

x : óháð breyta

y(x) : háðbreyta, háð x-i

$$y'(x) = \frac{dy}{dx}$$

y(x) er hnikað til að leita að lausn

Ef við leitum að lágmarki þá munu öll föll y(x) geta hærra gildi fyrir J en rétta lausnin

"Öll möguleg föll $y = y(\alpha, x)$ þ.a.

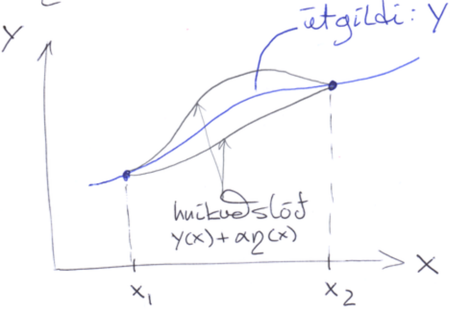
$$y(\alpha, x) = y(0, x) + \alpha\, \eta(x)$$

þ.s. $y(0,x)$ er fallið sem við erum að leita að

$y(0,x) = y(x)$

$\eta(x_1) = 0$ $\quad$ η er með samfellda 1. afleiðu

$\eta(x_2) = 0$



J verður fall af hnikunarstikanum α

$$\boxed{J(\alpha) = \int_{x_1}^{x_2} dx\, f\{y(\alpha, x), y'(\alpha, x); x\}}$$

þá er nauðsynlegt að

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$$

þýðir líka að J er ekki fall af α', en hærri veldi koma fyrir

til þess að J hafi útgildi fyrir $\alpha = 0$

óháð breyta, óháð α

skoðum

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} dx\, f\{y, y'; x\} = \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right]$$

borið saman við stikunina

$$y(\alpha, x) = y(x) + \alpha\eta(x) \longrightarrow \frac{\partial y}{\partial \alpha} = \eta(x), \quad \frac{\partial y'}{\partial \alpha} = \eta' = \frac{d\eta}{dx}$$

$$\rightarrow \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right]$$

hlutheildum

$$\int u\,dv = uv - \int v\,du$$

$$\left.\frac{\partial f}{\partial y'}\eta(x)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx$$

$$\rightarrow \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y} \eta(x) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \eta(x) \right]$$

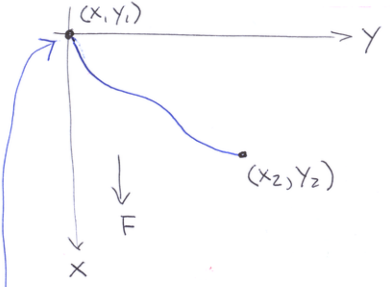
$$= \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right] \eta(x)$$

y og y' eru enn föll af α, $\eta(x)$ er hvaða fall sem er með viss stilgyrði

$$\rightarrow \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

<u>ákvarðar f</u> <u>Jafna Eulers</u> (1744)

Brachistochrone (stystur tími)



Ögn í föstu kraftsviði

Kyrr upphaflega í (x_1, y_1)

Getum við fundið lögun brautar milli (x_1, y_1) og (x_2, y_2) sem gæfi stystan ferðatíma?

Engin mótstaða, orkuvarðveitt

$\to T + U = \text{fasti}$

$T = 0$, og veljum $U = 0$

Núðar $T = \frac{1}{2} m v^2$ og $U = -mgx$

þyngdarkraftur

$\to T + U = 0$

$\to \frac{1}{2} v^2 = gx$

eða $v = \sqrt{2gx}$

$$\frac{ds}{dt} = v \qquad \frac{dy}{dx} = y'$$

$$t = \int_{(x_1 y_1)}^{(x_2 y_2)} \frac{ds}{v} = \int_{(x_1 y_1)}^{(x_2 y_2)} \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gx}} = \int_{x_1=0}^{x_2} \sqrt{\frac{1+y'^2}{2gx}}\, dx$$

fastinn $\frac{1}{\sqrt{2g}}$ skiptir ekki máli

við viljum lágmarka tímann

Fallið sem á að hámarka er $f(y';x) = \sqrt{\frac{1+y'^2}{x}}$

Jafna Eulers var

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

til að lágmarka heildið

hér $-\frac{d}{dx}\frac{\partial f}{\partial y'} = 0$

eða $\frac{\partial f}{\partial y'} = \text{fasti} = \frac{1}{\sqrt{2a}}$

þar sem a er nýr fasti

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{2a}} \quad \text{eða} \quad \frac{y'^2}{x(1+y'^2)} = \frac{1}{2a}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x}{2a}\left[1+\left(\frac{dy}{dx}\right)^2\right] \to \left(\frac{dy}{dx}\right)^2 = \frac{x}{\{2a - x\}} = \frac{x^2}{\{2ax-x^2\}}$$

$$\to dy = \frac{x\,dx}{\sqrt{2ax-x^2}} \quad \text{eða} \quad y = \int \frac{x\,dx}{\sqrt{2ax-x^2}}$$

Skiptum á breytu

$$x = a(1-\cos\theta) \to dx = a\sin\theta\,d\theta$$

$$y = \int \frac{a(1-\cos\theta)\,a\sin\theta\,d\theta}{\sqrt{2a^2(1-\cos\theta) - \{a(1-\cos\theta)\}^2}} = \int \frac{a(1-\cos\theta)\,a\sin\theta\,d\theta}{\sqrt{a^2(1-\cos^2\theta)}}$$

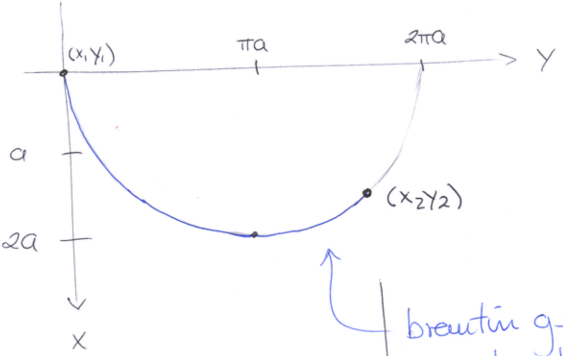
$$= \int a\{1-\cos\theta\}\,d\theta = a\{\theta - \sin\theta\} + C$$

← heildunarfasti

Stikuðu jöfnur hjólboga (Cycloid) eru

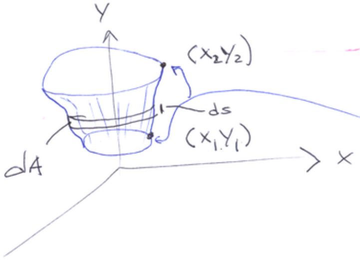
$$\boxed{\begin{aligned} x &= a(1-\cos\theta) \\ y &= a(\theta-\sin\theta) \end{aligned}}$$

því $C=0$ ef $(x_1, y_1) = (0\,0)$



brautin getur verið þ.a. lægsti punktur hennar liggi neðar en endapunkturinn $(x_2 y_2)$

Annað frægt dæmi



Tveimur punktum $(x_1 y_1)$ og $(x_2 y_2)$ snúið um y-ás – hvert er minnsta yfirborðið sem tengir hringina sem myndast?

sápuhimnur, ... (catenoid)

$$dA = 2\pi x\, ds = 2\pi x\sqrt{(dx)^2+(dy)^2}$$

$$y' = \frac{dy}{dx}$$

$$\rightarrow A = 2\pi\int_{x_1}^{x_2} dx\; x\sqrt{(1+y'^2)}$$

veljum $f = x\sqrt{1+y'^2}$ og reynum í $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{df}{dy'} = 0$

$\frac{\partial f}{\partial y} = 0$ en $\frac{\partial f}{\partial y'} = \frac{xy'}{\sqrt{1+y'^2}}$ $\rightarrow$ $\frac{d}{dx}\left\{\frac{xy'}{\sqrt{1+y'^2}}\right\} = 0$

$$\rightarrow \frac{xy'}{\sqrt{1+y'^2}} = a \text{ fasti}$$

$$\frac{xy'}{\sqrt{1+y'^2}} = a \;\rightarrow\; y'^2 = \left(\frac{a}{x}\right)^2\left(1+y'^2\right) \;\rightarrow\; \left\{1-\left(\frac{a}{x}\right)^2\right\} y'^2 = \left(\frac{a}{x}\right)^2$$

$$\rightarrow\; y' = \frac{a}{\sqrt{x^2-a^2}} \quad \text{eða} \quad y = \int \frac{a\,dx}{\sqrt{x^2-a^2}}$$

með lausn $y = a\,\mathrm{Arcosh}\left(\frac{x}{a}\right) + b$ ← heildunarfasti

eða

$$\boxed{x = a\cosh\left(\frac{y-b}{a}\right)}$$

lýsir catenoid yfirborði (keðjuyfirborð, keðjuferil)

lausn líka fyrir keðju hangandi milli tveggja punkta

Er lausnin altaf útgildi, lág- eða hámark?
sjá: Calculus of Variations
Robert Weinstock, Dover (1952)
Kafli 3-7, bls 30-31.

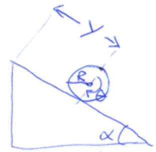
Nokkrar háðar breytur

Ef $f = f\{y_1(x), y_1'(x), y_2(x), y_2'(x), \ldots ; x\}$

þá fæst

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} \right) \eta_i(x)$$

$$\longrightarrow \quad \frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0, \quad i = 1, 2, \ldots, n$$



$y = R\theta$

$g(y;\theta) = y - R\theta = 0$

Velta

ytri skorður   lesa sjálf 6.6

Ef ytri skorður tengja breyturnar (háðu) $g_j\{y_i ; x\} = 0$

þá fæst

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) + \sum_j \lambda_j(x) \frac{\partial g_j}{\partial y_i} = 0}$$

Lagrange óákvarðaðir margfaldarar

skorður geta líka verið sem

$$\sum_i \frac{\partial g_i}{\partial y_i} dy_i = 0$$

Dæmi: föst lengd jaðars (isoperimetric)

$$J[y] = \int_a^b f\{y, y'; x\} dx \qquad \begin{aligned} y(a) &= A \\ y(b) &= B \end{aligned}$$

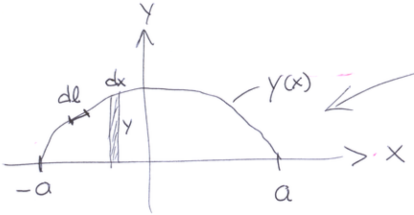
með skorðum

$$K[y] = \int_a^b g\{y, y'; x\} dx = \ell \leftarrow \text{lengd jaðars}$$

$y(x)$ er útgildi $\displaystyle\int_a^b (f + \lambda g) dx$

$$\rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right) = 0 \quad (*) \qquad \begin{aligned} y(a) &= A \\ y(b) &= B \\ K[y] &= \ell \end{aligned}$$

Verkefni Dídóar (ein útgáfa)



jaðar lengd ℓ

hvaða $y(x)$ gefur mestan flöt?

$$J = \int_{-a}^{a} y\,dx \qquad \begin{aligned} y(-a) &= 0 \\ y(a) &= 0 \end{aligned} \qquad K = \int_{-a}^{a} d\ell = \ell$$

$$K = \int_{-a}^{a} \sqrt{(dx)^2 + (dy)^2} = \int_{-a}^{a} dx \sqrt{1 + y'^2} = \ell \qquad \begin{aligned} y(x) &= y = f \\ g(x) &= \sqrt{1+y'^2} \end{aligned}$$

$$\frac{\partial f}{\partial y} = 1, \quad \frac{\partial f}{\partial y'} = 0, \quad \frac{\partial g}{\partial y} = 0, \quad \frac{\partial g}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$(*) \rightarrow \quad 1 - \lambda \frac{d}{dx}\left\{ \frac{y'}{\sqrt{1+y'^2}} \right\} = 0$$

$$\to \frac{d}{dx}\left\{\frac{y'}{\sqrt{1+y'^2}}\right\} = \frac{1}{\lambda} \quad \to \frac{\lambda y'}{\sqrt{1+y'^2}} = x + C_1$$

fasti

$$\to dy = \frac{\pm(x-C_1)\,dx}{\sqrt{\lambda^2-(x-C_1)^2}} \quad \to y = \mp\sqrt{\lambda^2-(x-C_1)^2} + C_2$$

$$\to (x-C_1)^2 + (y-C_2)^2 = \lambda^2$$

jaðar hrings

$C_1 = 0 \quad \lambda = a$ — vegna jaðarskilyrða
$C_2 = 0$

<u>δ-táknun</u> verkefni $\frac{\partial J}{\partial\alpha}d\alpha = \int_{x_1}^{x_2} dx\, d\alpha\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial\alpha}$

styttum skrift $\boxed{\delta J = \int_{x_1}^{x_2} dx\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\delta y}$

með

$$\boxed{\begin{aligned} \delta J &\equiv \frac{\partial J}{\partial \alpha} d\alpha \\ \delta y &\equiv \frac{\partial y}{\partial \alpha} d\alpha \end{aligned}}$$

útgildi

$$\delta J = \delta \int_{x_1}^{x_2} f\{y, y'; x\} dx = 0$$

Ef mörkin eru föst

$$\delta J = \int_{x_1}^{x_2} \delta f \, dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx$$

$$\delta y' = \delta\left(\frac{\partial y}{\partial x}\right) = \frac{d}{dx}(\delta y)$$

$$\rightarrow \delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \frac{d}{dx} \delta y \right) dx$$

hlutheildun

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right) \delta y \, dx$$