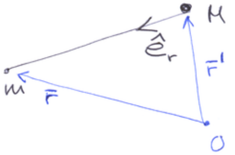
Þyngdarfræði

Newton: milli tveggja punktmassa er aðdráttarkraftur

$$\bar F = -G\,\frac{mM}{r^2}\,\hat e_r$$



$G = 6{,}673 \pm 0{,}010 \cdot 10^{-11}\ \mathrm{Nm}\ \mathrm{m^2/kg^2}$

Ef hluturinn með massa M er með samfellda dreifingu

$$\longrightarrow\ M = \int dv\,\rho(\bar r)$$

og

$$\underbrace{\bar F(\bar r)}_{\text{Þyngdarkraftur}} = -Gm\int_V \frac{\rho(\bar r')}{|\bar r-\bar r'|^2}\,\underbrace{\frac{\bar r-\bar r'}{|\bar r-\bar r'|}}_{\text{einingarvigur}}\,d^3r'$$

$\bar r$: Athugandi

$\bar r'$: uppspretta

Þyngdarsvið

$$\bar{g} \equiv \frac{\bar{F}}{m} = -G\frac{M}{r^2}\hat{e}_r$$

eða

$$\bar{g}(\bar{r}) = -G\int_V d\bar{r}'^3 \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|^2}\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|}$$

Þyngdarsviðið er sambærilegt við rafsvið í rafstöðufræði

það er geymið

$$\bar{\nabla}\times\bar{g} = 0$$

eða

$$\oint_C d\bar{l}\cdot\bar{g} = 0$$

Til gamans: Hve langt nær samanburður á þyngdarfræði og rafsegulfræði ?

Maxwell

$$\overline{\nabla}\cdot\overline{E} = \frac{\rho}{\epsilon_0}$$

$$\overline{\nabla}\cdot\overline{B} = 0$$

$$\overline{\nabla}\times\overline{E} = -\frac{\partial}{\partial t}\overline{B}$$

$$\overline{\nabla}\times\overline{H} = \overline{J} + \frac{\partial}{\partial t}\overline{D}$$

$$\overline{D} = \epsilon\overline{E}$$

$$\overline{H} = \frac{1}{\mu}\overline{B}$$

$$\overline{F}_e = q(\overline{E} + \overline{v}\times\overline{B})$$

Almennu sviðsjöfnur Einsteins -> gerðar línulegar

$$\overline{F}_g = m(\overline{g} + 4\overline{v}\times\overline{b})$$

$$\overline{\nabla}\cdot\overline{g} = -4\pi G\rho$$

$$\overline{\nabla}\cdot\overline{b} = 0$$

$$\overline{\nabla}\times\overline{g} = -\frac{\partial}{\partial t}\overline{b}$$

$$\overline{\nabla}\times\overline{b} = -\frac{4\pi G}{c^2}\overline{J}_g + \frac{1}{c^2}\frac{\partial}{\partial t}\overline{g}$$

segulhluti þyngdarsviðs

GPS

þyngdarbylgjur ?

En við höldum okkur við stöðufræðina

Þyngdarmætti

Samanburður við rafstöðufræðina, geymnir kraftar....

Til er mætti þ.a.

$$\bar{g} = -\bar{\nabla}\Phi$$

Vídd

$$[\bar{g}] = [\bar{a}] = \frac{L}{T^2} \rightarrow [\Phi] = \frac{L^2}{T^2} \quad \left([\nabla] = \frac{1}{L}\right)$$

$$\rightarrow [m\Phi] = M\frac{L^2}{T^2}$$: vídd orku

Fyrir punktmassa

$$\bar{g} = GM\frac{\hat{e}_r}{r^2}$$

ef punktmassinn er í miðju hnitakerfisins

$$\bar{g} = -\bar{\nabla}\Phi \rightarrow \Phi = -GM\frac{1}{r}$$

← sleppum heildunarfasta
Munum ávallt hafa áhuga á $\Delta\Phi$

$$\bar{\Phi}(\bar{r}) = -G\int_V d\bar{r}'^3 \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}$$ fyrir samfellda massadreifingu

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Vinna ytri krafts / massa til að flytja hann um $d\bar{r}$

$$dw' = -\bar{g}\cdot d\bar{r} = (\bar{\nabla}\bar{\Phi})\cdot d\bar{r} = \sum_i \frac{\partial\bar{\Phi}}{\partial x_i}dx_i = d\bar{\Phi}$$

Stöðuorka

$$U = m\bar{\Phi} \qquad (U = qV)$$

$$\rightarrow \qquad \bar{F} = -\bar{\nabla}U$$

Lögmál Gauß og jafna Poissons

Í bókinni er leitt út lögmál Gauß

$$\boxed{\bar{\nabla}\cdot\bar{g} = -4\pi G\rho} \quad \text{eða} \quad \boxed{\oint_S \bar{g}\cdot d\bar{s} = -4\pi G M}$$

Áður var komið $\bar{g} = -\bar{\nabla}\Phi$

$$\rightarrow \bar{\nabla}\cdot(\bar{\nabla}\Phi) = 4\pi G\rho \quad \text{eða} \quad \boxed{\nabla^2\Phi = 4\pi G\rho}$$

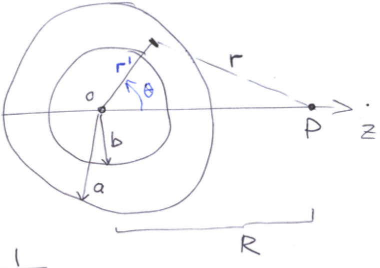
jafna Poissons gefur Φ ef ρ er þekkt

skalar jafna og því oft mjög handhæg í stað vigurjöfnu,
en lausnaraðferðir þarf að æfast og lærast G-IV. . . . .

Dæmi: Þyngdarmættið um og í kúluskel

Notum

$$\Phi(\bar{r}) = -G \int_V d\bar{r}'^3 \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}$$



Setjum miðju hnitakerfis í miðja skel og notum ϕ samhverfu

$$\Phi(R,0,0) = -2\pi\rho G \int_b^a r'^2 dr' \int_0^\pi \sin\theta \, d\theta \frac{1}{r}$$

$$r^2 = r'^2 + R^2 - 2r'R\cos\theta$$

$$\rightarrow 2r\,dr = 2r'R\sin\theta\, d\theta \longrightarrow \sin\theta\, d\theta \frac{1}{r} = \frac{dr}{r'R}$$

$$\rightarrow \Phi(R,0,0) = -\frac{2\pi\rho G}{R} \int_b^a r' dr' \int_{r_{min}}^{r_{max}} dr$$

Utan skeljar $R > a$

$$\Phi(R,0,0) = -\frac{2\pi\rho G}{R}\int_b^a r'dr' \int_{R-r'}^{R+r'} dr \qquad (R+r'-R+r') = 2r'$$

$$= -\frac{4\pi\rho G}{R}\int_b^a r'^2 dr' = -\frac{4}{3}\frac{\pi\rho G}{R}(a^3-b^3)$$

Massi skeljar $M = \frac{4\pi}{3}(a^3-b^3)\rho \rightarrow \boxed{\Phi(R,0,0) = -\frac{GM}{R} \quad R>a}$

utan skeljar er niðurstaðan óháð massadreifingu svo fremi hún sé óháð ϕ og θ

↑ má sjá með lögmáli Gauß

staðsetning P m.t.t. ϕ og θ á engu að breyta, valið til þæginda

Innan skeljar: $R < b$

$$\Phi(R,0,0) = -\frac{2\pi\rho G}{R}\int_b^a r'dr' \int_{r_{min}=r'-R}^{r_{max}=r'+R} dr \qquad (r'+R-r'+R)$$

$$= -4\pi\rho G\int_b^a r'dr' = -2\pi\rho G(a^2-b^2)$$

fasti óháður

R innan skeljar

Inni í skel: $b < R < a$

$$\Phi(R,0,0) = -\frac{2\pi\rho G}{R}\int_b^a r'dr' \left[\int_R^{r'+R} dr + \int_{R-r'}^{R} dr\right]$$

$$= -4\pi\rho G\left\{\frac{a^2}{2} - \frac{b^3}{3R} - \frac{R^2}{6}\right\}$$

Reynum aðra aðferð

Notum kúluhnit (r, ϕ, θ) með miðju í miðju skeljar

Gætum notað Gauß og reiknað þyngdarsviðið $\bar{g}$, en við höfum

$$\boxed{\nabla^2 \Phi = 4\pi G \rho}$$

ρ er fasti óháður ϕ og θ, skelin fellur að kúluhnitunum
$\to$ Φ er líka óháð ϕ og θ. Í kúluhnitum er jafnan því

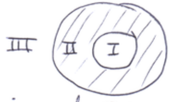
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{ r^2 \frac{\partial}{\partial r}\Phi(r)\right\} = 4\pi G \rho$$

Ein breyta eftir. Skrifum því

$$\frac{d}{dr}\left\{ r^2 \frac{d}{dr}\Phi(r)\right\} = 4\pi r^2 G \rho$$

heildum á ákveðið á þremur svæðum

$\rho \neq 0$ aðeins á II



Krefjumst samfelldni Φ

I :

$$\frac{d}{dr}\left\{r^2 \frac{d}{dr}\Phi(r)\right\} = 0 \qquad \underline{r < b}$$

aðeins möguleg lausn fyrir punkt hleðslu $\to C_1 = 0$ í I

$$\to r^2 \frac{d}{dr}\Phi(r) = C_1 \quad : \text{fasti}$$

$$\to \frac{d}{dr}\Phi(r) = \frac{C_1}{r^2} \quad \to \Phi_I(r) = -\frac{C_1}{r} + C_2 = C_2$$

III samskonar lausn fæst fyrir $\underline{r > a}$, en með öðrum föstum

$$\Phi_{III}(r) = -\frac{C_3}{r} + C_4$$

II

$$\frac{d}{dr}\left\{r^2 \frac{d}{dr}\Phi(r)\right\} = 4\pi r^2 G\rho \qquad b < r < a$$

$$\to r^2 \frac{d}{dr}\Phi(r) = \frac{4\pi}{3} r^3 G\rho + C_5, \quad \frac{d}{dr}\Phi(r) = \frac{4\pi}{3} r G\rho + \frac{C_5}{r^2}$$

$$\to \Phi_{II}(r) = \frac{4\pi}{6} r^2 G\rho - \frac{C_5}{r} + C_6$$

Skeytum saman í $r=a$, $r=b$

Veljum að $\Phi_{III}(r \to \infty) = 0 \to \underline{C_4 = 0}$

$\underline{r=a}$ $\quad \Phi_{III}(a) = \Phi_{II}(a) \to \boxed{-\frac{C_3}{a} = \frac{4\pi a^2 G\rho}{6} - \frac{C_5}{a} + C_6}$

$\underline{r=b}$ $\quad \Phi_{I}(b) = \Phi_{II}(b) \to \boxed{C_2 = \frac{4\pi b^2 G\rho}{6} - \frac{C_5}{b} + C_6}$

4 fastar, tvær jöfnur, þurfum líka samfellu $\bar{g} = -\bar{\nabla}\Phi$

$\bar{g}$ er aðeins með útþátt (radial) $\bar{g}\cdot\hat{e}_r = -\frac{\partial}{\partial r}\Phi$

$\underline{r=a}$ $\quad \Phi'_{III}(a) = \Phi'_{II}(a) \to \boxed{\frac{C_3}{a^2} = \frac{4\pi a}{3} G\rho + \frac{C_5}{a^2}}$

$\underline{r=b}$ $\quad \Phi'_{I}(b) = \Phi'_{II}(b) \to \boxed{0 = \frac{4\pi b G\rho}{3} + \frac{C_5}{b^2}}$

$$-C_3 + C_5 - aC_6 = \frac{4\pi a^3 G\rho}{6}$$

$$C_2 + \frac{C_5}{b} - C_6 = \frac{4\pi b^2 G\rho}{6}$$

$$C_3 - C_5 = \frac{4\pi a^3 G\rho}{3}$$

$$-C_5 = \frac{4\pi b^3 G\rho}{3}$$

$$C_5 = -\frac{4\pi b^3 G\rho}{3}$$

$$C_3 = \frac{4\pi G\rho}{3}\left\{a^3 - b^3\right\} = M$$

$$-aC_6 = \frac{4\pi a^3 G\rho}{6} + C_3 - C_5 = \frac{4\pi G\rho}{6}\left\{a^3 + 2a^3\right\} = 2\pi G\rho a^3$$

$$\to C_6 = -2\pi G\rho a^2$$

$$C_2 = \frac{4\pi b^2 G\rho}{6} + C_6 - \frac{C_5}{b} = \frac{4\pi G\rho}{6}\left\{b^2 - \frac{6}{2}a^2 + 2b^2\right\}$$

$$= 2\pi G\rho\left\{b^2 - a^2\right\}$$

því er lausnin

$$\begin{cases} \Phi_{I}(r) = -2\pi G\rho\left\{a^2 - b^2\right\} & \underline{r < b} \\ \Phi_{II}(r) = -4\pi G\rho\left\{-\frac{r^2}{6} - \frac{b^3}{3r} + \frac{a^2}{2}\right\} & \underline{b < r < a} \\ \Phi_{III}(r) = -\frac{4\pi G}{3r}\rho\left\{a^3 - b^3\right\} = -\frac{GM}{r} & \underline{r > a} \end{cases}$$

og því

$$\bar{g}_{I}(r) = 0$$

$$\bar{g}_{II}(r) = \frac{4\pi\rho G}{3}\left\{\frac{b^3}{r^2} - r\right\}\hat{e}_r$$

$$g_{III}(r) = -\frac{GM}{r^2}\hat{e}_r$$