Ólínulegar sveiflur og ringl

Þvingaða og deyfða sveiflunum var lýst með hreyfijöfnu sem er eitt tilfelli af almennari jöfnum

$$\boxed{m\ddot{x} + f(\dot{x}) + g(x) = h(t)}$$

$f(\dot{x})$ og $g(x)$ geta verið ólínuleg föll, ef svo þá eru

> ekki til almennar lausnar-aðferðir fyrir greini reikninga

> Tölulegar lausnir

P.S. de Laplace ↔ framreikn.

Henri Poincaré (1854-1912)

→ Ringl (Chaos)

Tölvur – tölulegar lausnir

→ Fermi-Pasta-Ulam (1953)

→ 1970 – 1980 . . . .

Næmni á upphafsástand

⋮

skammtafræði

Ólínulegar sveiflur

Við þekkjum hreintóna-sveiflurnar sem fást í mættinu

$$U(x) = \frac{1}{2}kx^2$$

og kraft þess

$$F(x) = -kx$$

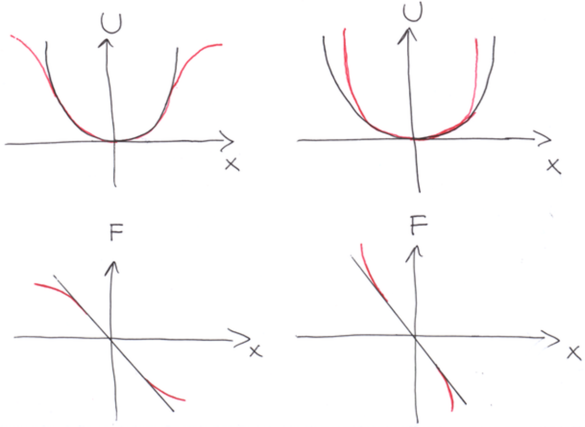
Þetta er oft nálgun við raunkerfi

innilokun

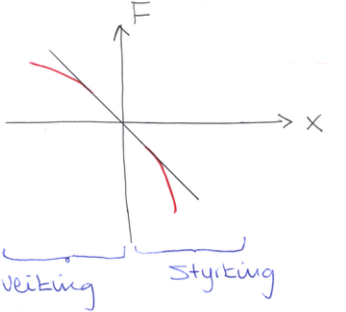
Mörg kerfi sýna veikingu eða styrkingu innilokunar

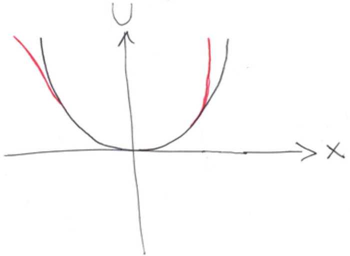
$$F(x) \simeq -kx \mp \varepsilon x^3$$

$$U(x) \simeq \frac{1}{2}kx^2 \pm \frac{1}{4}\varepsilon x^4$$

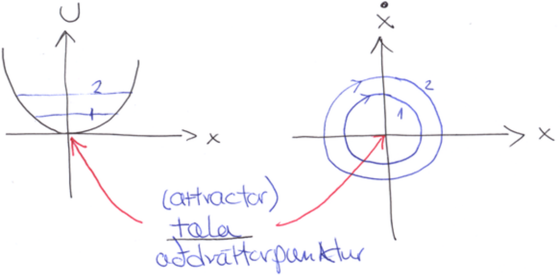


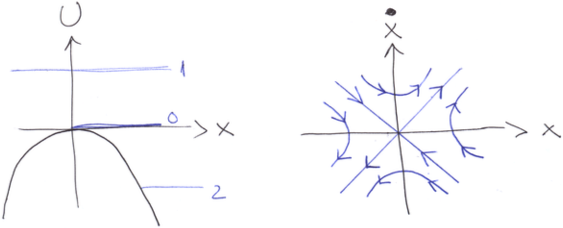
eða ósamhverft





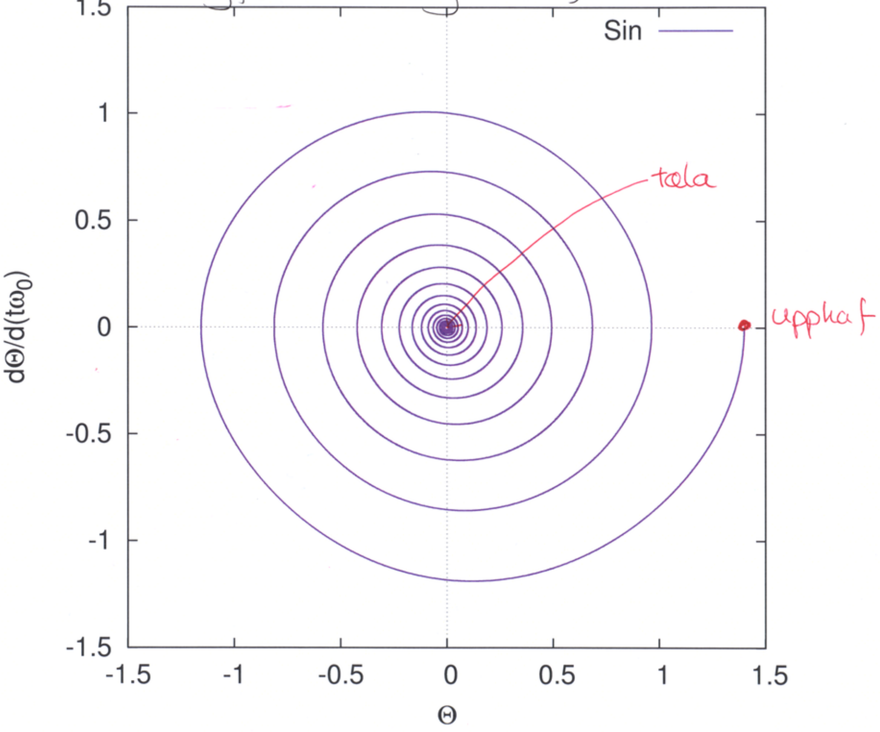
Heppilegt að skoða fasarit



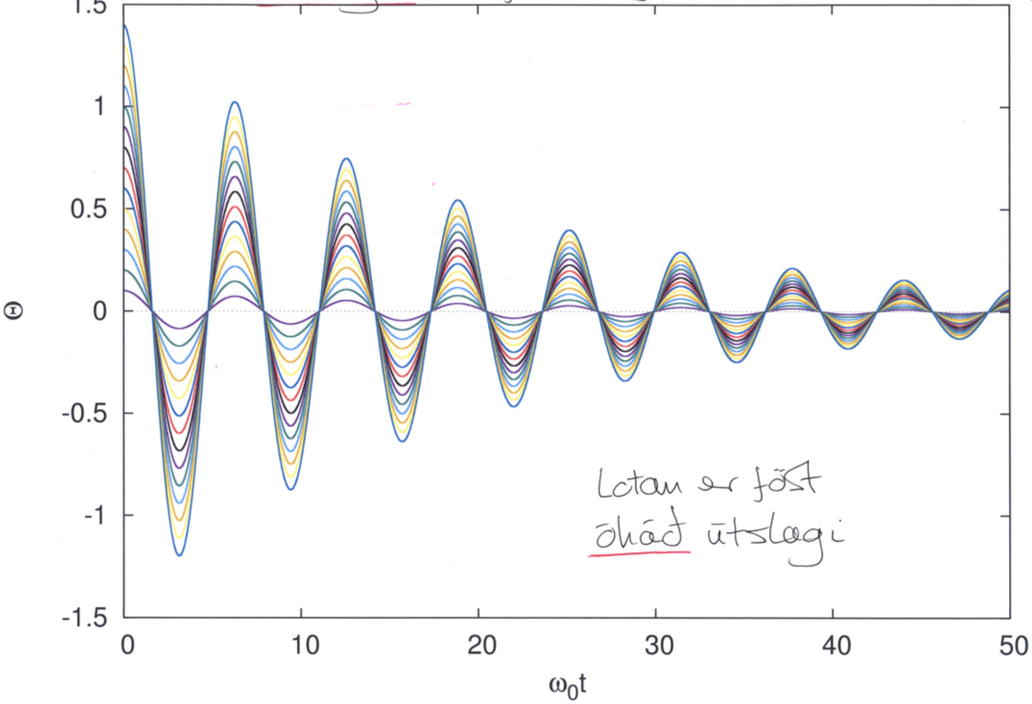


(Fola ?) Ferilskil (separatrix)

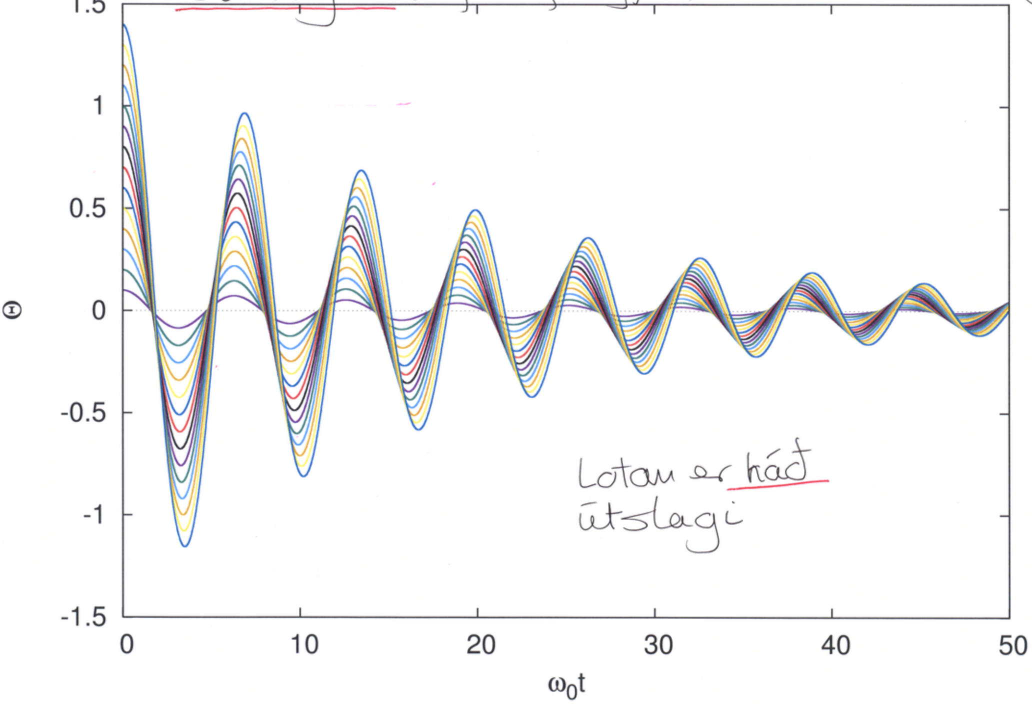
Deyfður ólínulegur sveifill



línulegur sveifill, deyfður



'Ólínulegur sveifill, deyfður



Lotan er háð útslagi

Ólínulegi sveifill van der Pols

B. van der Pol skoðaði ólínulegar sveiflur í rás með útvarpslampa. Jafnan sem lýsti þeim er

$$\boxed{\ddot{x} + \mu(x^2 - a^2)\dot{x} + \omega_0^2 x = 0}$$

μ er jákvæður fasti (lítill, en það hefur ekki merkingu nema við gerum hann víddarlausann eða berum saman við aðrar stærðir).

Útslag $|x|$ miðað við $|a|$ ræður því hvort $\dot{x}$-liðurinn sé deyfing eða styrking skoðum tölulega lausn, en fyrst þarf að stala jöfnuna til

$$\ddot{x} + \mu(x^2 - a^2)\dot{x} + \omega_0^2 x = 0$$

En, tíminn er enn að flækjast fyrir

$$\frac{\ddot{x}}{a} + \mu a^2\left(\left(\frac{x}{a}\right)^2 - 1\right)\frac{\dot{x}}{a} + \omega_0^2 \frac{x}{a} = 0$$

Setjum $t \to \omega_0 t$ ← víddarlaust og $\frac{df}{d(t\omega_0)} = f''$, $z = \frac{x}{a}$

$s = t\omega_0$

$$\omega_0^2 z'' + (\mu a^2 \omega_0)\left\{z^2 - 1\right\} z' + \omega_0^2 z = 0$$

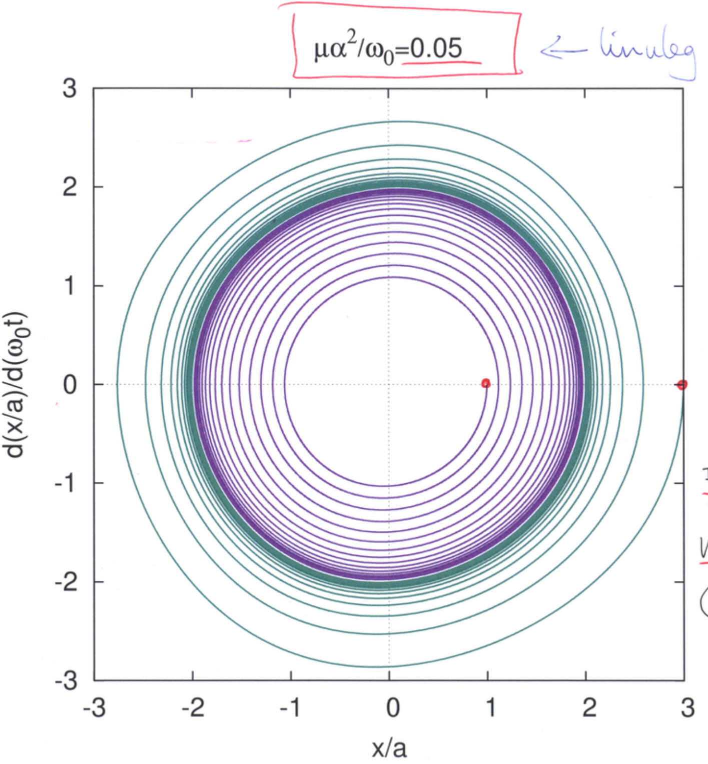
eða

$$\boxed{z'' + \left(\frac{\mu a^2}{\omega_0}\right)(z^2 - 1)\, z' + z = 0}$$

Nú sést hvað er lítið, allir liðir eru víddarlausir hér

$[z] = 1$, $[s] = 1$ $\left[\frac{\mu a^2}{\omega_0}\right] = 1$

óháð því hvaða einingu x hafði í upphafi



← linuleg hegðun .....

Tveir mism. upphafspunktar

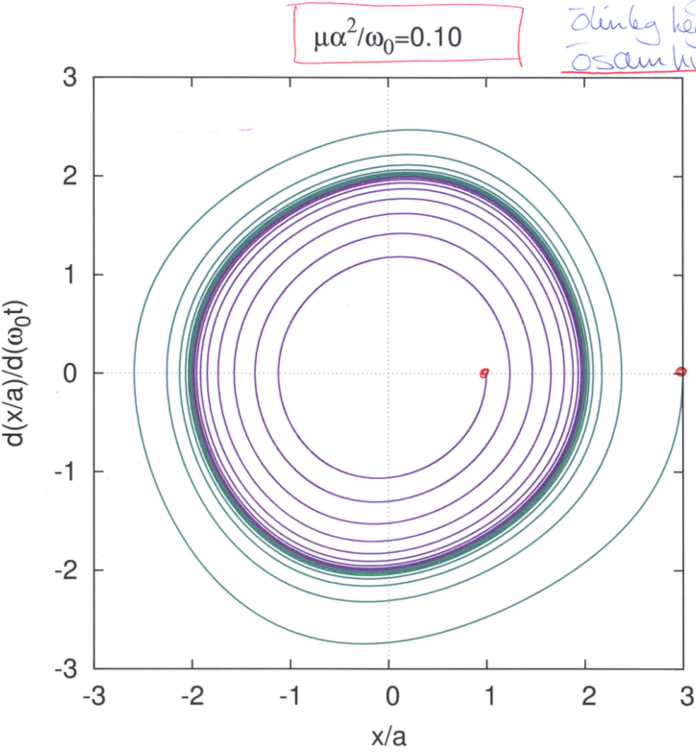
Engin tala,

mark ferill

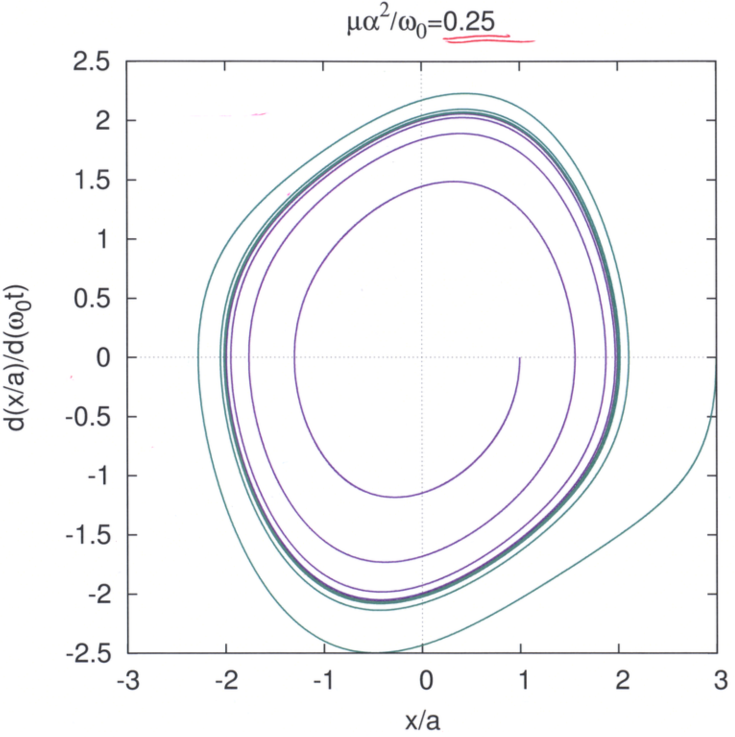
(limit cycle)

10

ólínul. hegðun byrjar ósamhverfa

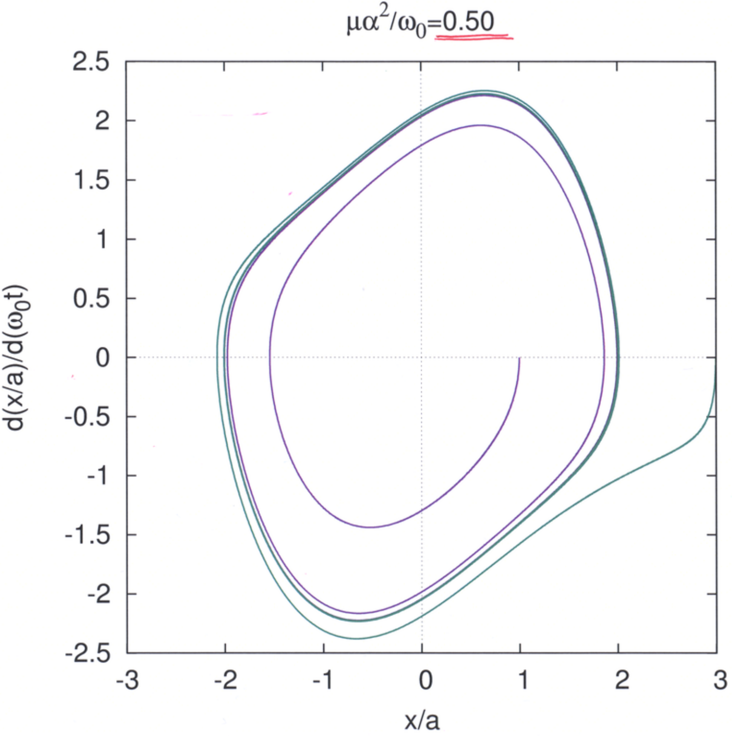


upphafsp.



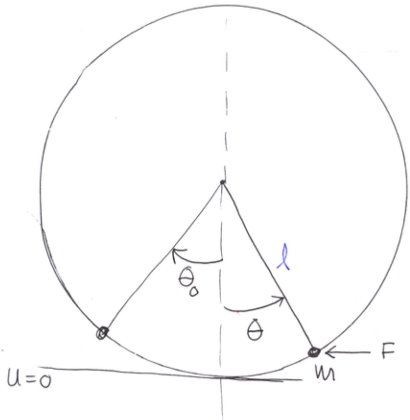
$\mu\alpha^2/\omega_0$=0.25
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2
-1
0
1
2
3
d(x/a)/d(ω_0t)
x/a

12



$\mu\alpha^2/\omega_0=0.50$
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2
-1
0
1
2
3
x/a
d(x/a)/d(ω_0t)

Sveifill



Hornið θ þarf ekki að vera lítið

$$I\ddot{\theta} = lF \quad , \quad I = ml^2 \text{ hverfitregða}$$

$$F = -mg\sin\theta$$

Fyrir lítið horn er krafturinn línulegur þá er sveifillinn hreintóna

$$\ddot{\theta} + \omega_0^2\theta = 0$$

Annars er hreyfijafnan

$$\boxed{\ddot{\theta} + \omega_0^2\sin\theta = 0}$$

Lausnina má skrifa á lokuðu formi með sporbaugsföllum, en við sleppum því →

Reynum að reikna, með tímaflanareikn.

Geymið Kerfi $\longrightarrow$ $\underline{T + U = E \text{ fasti}}$

$$T = \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} m \ell^2 \dot{\theta}^2$$

$$U = mg\ell(1 - \cos\theta)$$

$$T(\theta_0) = 0 \quad \text{upphafsparkstur}$$

$$U(\theta_0) = mg\ell(1 - \cos\theta_0)$$

$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right) \rightarrow E = U(\theta_0) = 2mg\ell \sin^2\left(\frac{\theta_0}{2}\right)$$

og $U = 2mg\ell \sin^2\left(\frac{\theta}{2}\right)$

$$T = E - U = 2mg\ell\left\{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right\} = \frac{1}{2} m\ell^2\dot{\theta}^2$$

$$\rightarrow \dot{\theta} = 2\sqrt{\frac{g}{\ell}}\sqrt{\underbrace{\left\{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right\}}_{\geq 0}} = \frac{d\theta}{dt}$$

Til að nálgast lotuna nýtum við

$$dt = \frac{\frac{1}{2}\sqrt{\frac{l}{g}}\, d\theta}{\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}$$

$$\boxed{\frac{\tau}{4} = \frac{1}{2}\sqrt{\frac{l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}}$$

venja er að gera breytuskipti

$$z = \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_0}{2}\right)}, \quad k = \sin\left(\frac{\theta_0}{2}\right)$$

$$\rightarrow dz = \frac{\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta_0}{2}\right)} d\theta = \frac{\sqrt{1-k^2z^2}}{2k} d\theta$$

$$\longrightarrow \boxed{\tau = 4\sqrt{\frac{l}{g}}\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}}$$

nákvæm lausn

τ er háð θ_0 í gegnum k

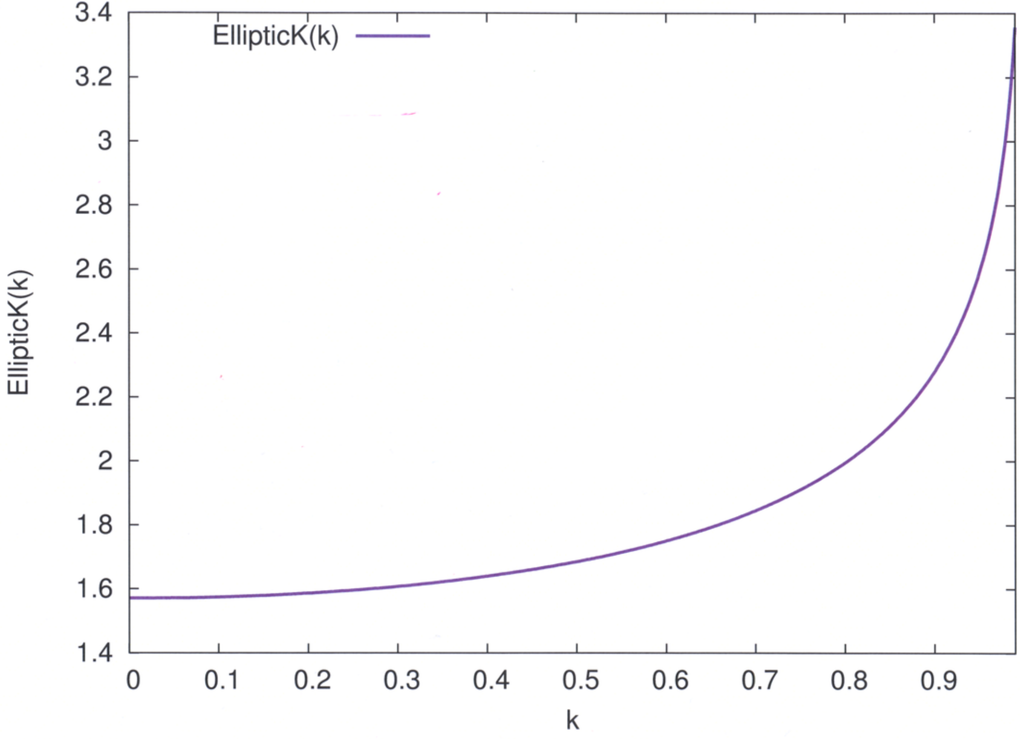
Vitum

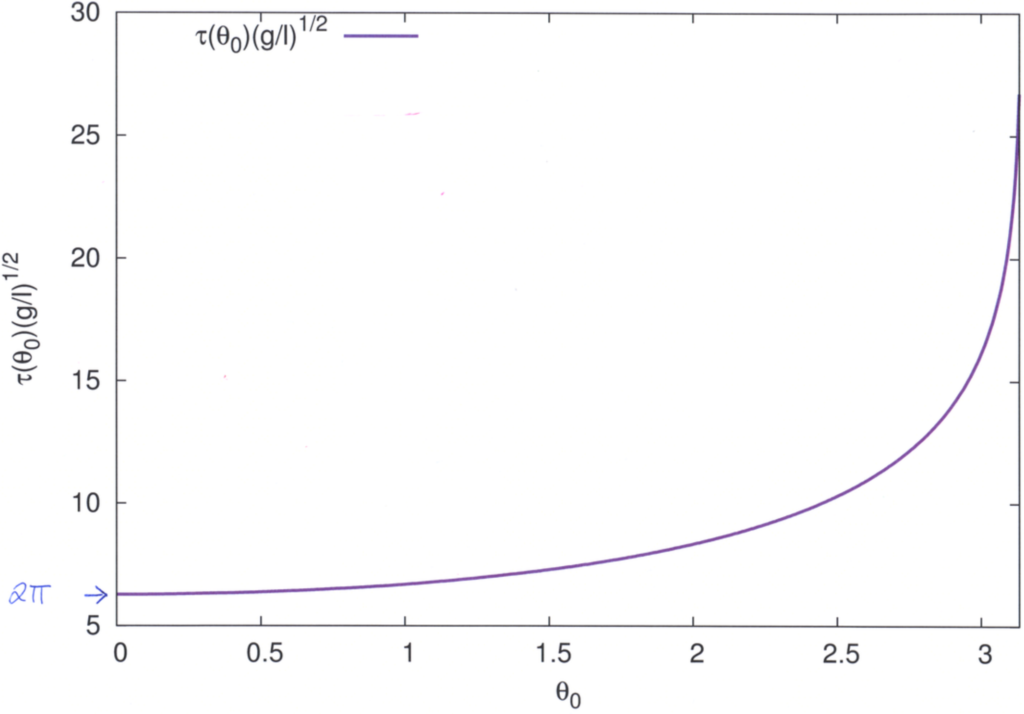
$$\left(1-(kz)^2\right)^{-1/2} = 1 + \frac{(kz)^2}{2} + \frac{3}{8}(kz)^4 + \cdots\cdots, \quad kz < 1$$

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 \frac{dz}{\sqrt{1-z^2}}\left\{1 + \frac{(kz)^2}{2} + \frac{3}{8}(kz)^4 + \cdots\right\}$$

$$= 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{k^2}{4} + \frac{9}{64}k^4 + \cdots\right\} \qquad k = \sin\left(\frac{\theta_0}{2}\right)$$

$$\tau = 4\sqrt{\frac{l}{g}}\,F(k,1)$$


EllipticK(k)
EllipticK(k)
k
3.4
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9



30
25
20
15
10
5
τ(θ0)(g/l)1/2
0
0.5
1
1.5
2
2.5
3
θ0
2π